Varentropy of Stable Laws: A Constructive Formula at Rational Index

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Abstract

This article gives a constructive formula for the varentropy of non-extremal one-dimensional stable laws with rational stability index $\alpha = p/q$, $\alpha \neq 1$. The stable density is first decomposed into its positive and negative half-line pieces and each half-line piece is represented by a Fox–H kernel. For rational α, these kernels split into finite hypergeometric residue classes and are holonomic. We then introduce, for each half-line kernel, an endpoint-matched hyperexponential factor

$$B(z) = H_0(1 + \kappa z^p)^{-\nu},$$

chosen to match both the finite endpoint value and the leading algebraic tail. The factor contribution to varentropy has exact special-function moments expressible by ordinary generalized hypergeometric functions together with explicit Pochhammer–polygamma sums. The normalized residual N satisfies $N(0+) = N(\infty) = 1$; the remaining logarithmic residual moments are D-algebraic periods evaluated on a zero-free dyadic logit atlas. We specify the branch, radius, coefficient, truncation, and tail certificates needed for a certified computation. We also record a varentropy convergence theorem in the non-extremal stable central limit theorem, extracted from the Bobkov–Chistyakov–Götze framework.

Keywords: self-information, varentropy, entropy, risk management, heavy-tailed distributions, stable laws

Contents

I Introduction

This paper is the second paper in a series whose aim is to transform varentropy into a practical tool for heavy-tailed risk models. We omit the general motivation for varentropy and focus on the stable distributions. Please see [1], [2] if you are interested in motivation. Due to the ubiquitous position of the stable laws as limiting distributions of normalized sums of independent identically distributed random variables, it would be quite difficult to talk about practical aspects of varentropy without addressing the varentropy of stable laws first.

If X has density f, its varentropy is

$$\mathrm{VE}(X) = \mathrm{Var}[-\log f(X)].$$

For a normalized density this is the second derivative at $\lambda = 1$ of the logarithm of the Renyi partition function

$$Z_f(\lambda) = \int_{\mathbb{R}} f(x)^{\lambda}\ \mathrm{d}x, \qquad \mathrm{VE}(X) = \left.\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \log Z_f(\lambda)\right|_{\lambda=1}.$$

Stable densities are not elementary in general, but they admit Fox–H and Mellin–Barnes descriptions. The density-level curvature terms

$$f \log f, \qquad f(\log f)^2$$

are the objects whose integrals determine varentropy. Our strategy is to split the density into positive and negative half-line pieces, express each piece through a Fox–H kernel, and then factor each half-line kernel into an exactly integrable hyperexponential part and a normalized residual.

For rational $\alpha = p/q$, the half-line Fox–H kernels reduce to finite sums of generalized hypergeometric functions and are therefore holonomic. Their logarithmic curvature terms are generally not holonomic, but they are D-algebraic on zero-free analytic patches. Recent constructive work on D-algebraic functions and algorithms for nonlinear differential equations gives the right language for this residual class [3], [4].

The stable-density material in Section II is not claimed as new. It is a normalization of the Feller–Zolotarev phase chart, the standard Fox–H Mellin kernel, and the rational-index residue expansions of Hatzinikitas–Pachos, with constants chosen for the varentropy calculation. The new ingredients start in Section III: the scale-cancelled reduction of varentropy to half-line kernel power moments, the endpoint-matched hyperexponential factor, the exact factor-moment residue formulas, and the zero-free dyadic logit atlas for the residual D-algebraic periods.

The main formula is

$$\mathrm{VE}_{\alpha,\beta} = \frac{\mathcal{B}_2 + \mathcal{R}_2}{\alpha} - \left(\frac{\mathcal{B}_1 + \mathcal{R}_1}{\alpha}\right)^2,$$

where $\mathcal{B}_1, \mathcal{B}_2$ are exact moments of the endpoint-matched hyperexponential factors and $\mathcal{R}_1, \mathcal{R}_2$ are residual D-algebraic moments. All notation is introduced in Sections II and III.

Section II fixes the parameterization and records the known half-line kernel and rational residue formulas in the normalization used later. Section III contains the constructive varentropy formula and the residual-evaluation certificate. Section IV proves varentropy convergence in the non-extremal stable central limit theorem. The appendices collect the longer residue formulas, Laurent bookkeeping, and coefficient recurrences.

II Stable half-line kernels and rational residue classes

This section fixes notation and normalizations. The Fox–H kernel, its Mellin transform, and the rational-index hypergeometric residue classes are standard consequences of the Feller–Zolotarev phase representation and of the Hatzinikitas–Pachos rational stable-density formulas. We include short proofs only to make the constants, half-line masses, and residue conventions unambiguous for the varentropy calculation. The new use of these formulas begins in Section III.

II-A Parameterization and assumptions

We work with standardized one-dimensional stable laws. Location and scale are set to zero and one, since varentropy is invariant under affine transformations: if $Y = aX + c$, $a \neq 0$, then

$$-\log f_Y(Y) = -\log f_X(X) + \log |a|,$$

and adding a constant does not change variance.

For $0 < \alpha < 2$, $\alpha \neq 1$, the conventional skewness coordinate used in this paper is Nolan's S_1 characteristic-function convention with unit scale and zero location:

$$\varphi^{(1)}_{\alpha,\beta}(t) = \exp\left\{-|t|^{\alpha}\left(1 - \mathrm{i}\beta \tan\frac{\pi\alpha}{2}\ \operatorname{sgn} t\right)\right\}, \qquad -1 \le \beta \le 1. \tag{II.1}$$

This convention is used only for $\alpha \neq 1$ in the main rational-index formula. The skewed $\alpha = 1$ case is structurally different and is discussed separately in Subsection III-H.

For the Fox–H formula it is more convenient to use the normalized Feller phase b. Put

$$\theta = \frac{\pi\alpha b}{2}.$$

The branch convention is the following explicit one: θ is the unique real number in

$$-\theta_{\max}(\alpha) \le \theta \le \theta_{\max}(\alpha), \qquad \theta_{\max}(\alpha) = \frac{\pi}{2}\min(\alpha, 2-\alpha), \tag{II.2}$$

that satisfies

$$\tan\theta = \beta \tan\frac{\pi\alpha}{2}. \tag{II.3}$$

Equivalently,

$$\beta = \frac{\tan(\pi\alpha b/2)}{\tan(\pi\alpha/2)}, \qquad b = \frac{2}{\pi\alpha}\operatorname{arctan}_{\alpha}\left(\beta\tan\frac{\pi\alpha}{2}\right), \tag{II.4}$$

where $\operatorname{Arctan}_{\alpha}$ denotes the branch selected by (II.2). In practice this is the principal real arctangent with the value forced to lie in the admissible Feller interval. Thus

$$|b| \le b_{\max}(\alpha) := \min\left(1, \frac{2-\alpha}{\alpha}\right). \tag{II.5}$$

With this branch,

$$\varphi^{(1)}_{\alpha,\beta}(t) = \exp\{-A^{\alpha}|t|^{\alpha}e^{-\mathrm{i}\theta \operatorname{sgn} t}\}, \qquad A = \sec^{1/\alpha}\theta. \tag{II.6}$$

Hence the scale-one S_1 variable is the positive scale multiple A of the Feller-phase variable with characteristic function

$$\varphi^{F}_{\alpha,b}(t) = \exp\{-|t|^{\alpha}e^{-\mathrm{i}\pi\alpha b \operatorname{sgn}(t)/2}\}. \tag{II.7}$$

The scale factor A is harmless for varentropy, but it is retained in the density formula to keep the conventional S_1 normalization visible.

Nolan's S_0 convention is not the density convention used in (II.1). For $\alpha \neq 1$ it differs from S_1 by a deterministic location shift. With scale γ and location δ_0, one common S_0 form is

$$\exp\left\{\mathrm{i}\delta_0 t - |\gamma t|^{\alpha}\left[1 + \mathrm{i}\beta \operatorname{sgn}(t)\tan\frac{\pi\alpha}{2}\left(|\gamma t|^{1-\alpha} - 1\right)\right]\right\};$$

it corresponds to the S_1 location $\delta_1 = \delta_0 - \beta\gamma\tan(\pi\alpha/2)$ when $\alpha \neq 1$. Since varentropy is invariant under translations and positive rescalings, this distinction does not change the value of VE for a fixed (α, β) away from $\alpha = 1$, but all half-line kernels and the conversion (II.4) below are written in the S_1/Feller convention.

For each side $s \in \{+1, -1\}$ define

$$\rho_s = \frac{1 + sb}{2}. \tag{II.8}$$

The non-extremal two-sided case means $0 < \rho_+, \rho_- < 1$. The Gaussian endpoint is $\alpha = 2$, forcing $b = 0$.

II-B Fox–H half-line kernel

Let

$$A = A_{\alpha,b} := \sec^{1/\alpha}\Big(\frac{\pi\alpha b}{2}\Big). \tag{II.9}$$

For $x \neq 0$, write $s = \mathrm{sgn}(x)$ and $z = |x|/A > 0$. The standardized density is represented by

$$f_{\alpha,b}(sAz) = \frac{1}{\alpha A} K_{\alpha,\rho_s}(z), \qquad z > 0. \tag{II.10}$$

The Fox–H half-line kernel has Mellin–Barnes form

$$K_{\alpha,\rho}(z) = \frac{1}{\pi}\frac{1}{2\pi \mathrm{i}} \int_C \Gamma(r)\Gamma\Big(\frac{1-r}{\alpha}\Big)\sin\big(\pi\rho(1-r)\big) z^{-r}\,\mathrm{d}r. \tag{II.11}$$

This follows from the standard Fox–H representation and Euler's reflection formula; see Mathai–Saxena–Haubold for the general Fox–H theory [5].

Remark II.1. Hatzinikitas and Pachos give the asymmetric stable density in Fox–H form in [6] Eq. (38)-Eq.(40) Their parameter

$$\delta_{\mathrm{HP}} = 1 + \frac{2}{\alpha\pi}\arctan\Big(\beta\tan\frac{\pi\alpha}{2}\Big)$$

becomes, in our phase coordinate, $\delta_{\mathrm{HP}} = 1 + b = 2\rho_+$ on the positive half-line. The negative half-line is obtained by replacing b with $-b$, hence by using $2\rho_-$. If $H_{\mathrm{HP}}(z)$ denotes the Fox–H function appearing in [6] Eq. (38)-Eq.(40), then our half-line kernel is

$$K_{\alpha,\rho}(z) = \frac{H_{\mathrm{HP}}(z)}{z}.$$

Thus, after the division by z and the scale convention (II.9), the Hatzinikitas–Pachos formula is exactly (II.10).

Lemma II.2 (Half-line Mellin transform)**.** *For $0 < \rho < 1$, the Mellin transform of the half-line kernel is*

$$\mathcal{M}_{\alpha,\rho}(r) := \int_0^\infty z^{r-1} K_{\alpha,\rho}(z)\,\mathrm{d}z = \frac{1}{\pi}\Gamma(r)\Gamma\Big(\frac{1-r}{\alpha}\Big)\sin\big(\pi\rho(1-r)\big), \tag{II.12}$$

first in the common Mellin strip and then by analytic continuation.

Proof. Formula (II.11) is the inverse Mellin representation of the gamma product on the right of (II.12). Mellin inversion gives the transform in the fundamental strip, and meromorphic continuation gives the displayed identity elsewhere where finite values are obtained by removable singularities. ☐

Lemma II.3 (Half-line mass)**.** *For $0 < \rho < 1$,*

$$\int_0^\infty K_{\alpha,\rho}(z)\,\mathrm{d}z = \alpha\rho. \tag{II.13}$$

Consequently $\Pr(X > 0) = \rho_+$ *and* $\Pr(X < 0) = \rho_-$.

Proof. By Lemma II.2, the desired integral is the Mellin transform at $r = 1$:

$$\int_0^\infty K_{\alpha,\rho}(z)\,\mathrm{d}z = \mathcal{M}_{\alpha,\rho}(1),$$

provided we interpret the right-hand side by its removable value. Put $\varepsilon = 1 - r$. As $\varepsilon \to 0$,

$$\Gamma(r) = \Gamma(1-\varepsilon) = 1 + O(\varepsilon),$$
$$\Gamma\Big(\frac{1-r}{\alpha}\Big) = \Gamma\Big(\frac{\varepsilon}{\alpha}\Big) = \frac{\alpha}{\varepsilon} + O(1),$$

where we used the standard expansion $\Gamma(u) = u^{-1} + O(1)$ at $u = 0$, and

$$\sin\big(\pi\rho(1-r)\big) = \sin(\pi\rho\varepsilon) = \pi\rho\varepsilon + O(\varepsilon^3).$$

Substituting these three expansions into (II.12) gives

$$\mathcal{M}_{\alpha,\rho}(r) = \frac{1}{\pi}(1 + O(\varepsilon))\Big(\frac{\alpha}{\varepsilon} + O(1)\Big)(\pi\rho\varepsilon + O(\varepsilon^3)) = \alpha\rho + O(\varepsilon).$$

Hence the removable value is $\mathcal{M}_{\alpha,\rho}(1) = \alpha\rho$, proving (II.13).

Finally, on the positive half-line the change of variables $x = Az$ gives

$$\Pr(X > 0) = \int_0^\infty f_{\alpha,b}(x)\,\mathrm{d}x = \frac{1}{\alpha}\int_0^\infty K_{\alpha,\rho_+}(z)\,\mathrm{d}z = \rho_+.$$

The negative half-line uses $x = -Az$ and ρ_- in the same way, giving $\Pr(X < 0) = \rho_-$. ☐

II-C Exact density series

The Mellin integrand in (II.11) has pole families at $r = -j$ and at $r = 1 + \alpha k$. Which family gives a convergent real-axis series depends on the side of $\alpha = 1$.

Theorem II.4 (Central series, $1 < \alpha \le 2$). *For $1 < \alpha \le 2$ and $0 < \rho < 1$,*

$$K_{\alpha,\rho}(z) = \sum_{j=0}^{\infty} a_j(\alpha,\rho) z^j, \qquad a_j(\alpha,\rho) = \frac{(-1)^j}{\pi j!}\Gamma\left(\frac{j+1}{\alpha}\right)\sin\bigl(\pi\rho(j+1)\bigr). \tag{II.14}$$

The series is entire. This is the kernel form of the small-z expansion of Hatzinikitas–Pachos [6, Eq. (41)], after the substitution $K_{\alpha,\rho} = H_{\mathrm{HP}}/z$.

Proof. Close the Mellin–Barnes contour across the poles $r = -j$ of $\Gamma(r)$. The residue of $\Gamma(r)$ at $-j$ is $(-1)^j/j!$, and the remaining factors give exactly (II.14). Stirling's formula applied to $\Gamma((j+1)/\alpha)/j!$ gives infinite radius of convergence for $\alpha > 1$. □

Theorem II.5 (Tail series, $0 < \alpha < 1$). *For $0 < \alpha < 1$ and $0 < \rho < 1$,*

$$K_{\alpha,\rho}(z) = \sum_{k=1}^{\infty} c_k(\alpha,\rho) z^{-1-\alpha k}, \tag{II.15}$$

where

$$c_k(\alpha,\rho) = \frac{\alpha}{\pi}\frac{(-1)^{k+1}}{k!}\Gamma(1+\alpha k)\sin(\pi\alpha\rho k). \tag{II.16}$$

The series converges for every $z > 0$. This is the kernel form of the large-z expansion of Hatzinikitas–Pachos [6, Eq. (42)], again after dividing their Fox–H function by z.

Proof. Close the contour across the poles $r = 1 + \alpha k$, $k \ge 1$, of $\Gamma((1-r)/\alpha)$. The $k = 0$ pole gives no contribution because the sine factor vanishes. With $u = (1-r)/\alpha$, the residue at $u = -k$ contributes $(-1)^{k+1}\alpha/k!$. After including the remaining factors and orientation, one obtains (II.16). Stirling's formula shows convergence for $0 < \alpha < 1$. □

Proposition II.6 ($\alpha = 1$ transition kernel). *At $\alpha = 1$,*

$$K_{1,\rho}(z) = \frac{\sin(\pi\rho)}{\pi(1 + 2z\cos(\pi\rho) + z^2)}. \tag{II.17}$$

At $\rho = 1/2$ this is the Cauchy kernel $1/\{\pi(1+z^2)\}$.

Proof. At $\alpha = 1$ the two pole families coalesce. Summing the resulting geometric residue series gives

$$\frac{\sin(\pi\rho)}{\pi(1 + ze^{\mathrm{i}\pi\rho})(1 + ze^{-\mathrm{i}\pi\rho})},$$

which is (II.17). □

II-D Rational residue classes and holonomicity

The decisive symbolic simplification occurs when $\alpha = p/q$ is rational, $(p,q) = 1$. Gamma values at rational offsets become fixed constants and index shifts turn into Pochhammer products.

Theorem II.7 (Rational residue representation, central branch). *Assume $1 < \alpha = p/q \le 2$ with $p > q$. Put $\tau = e^{\mathrm{i}\pi\rho}$. For $r = 0, \dots, p-1$ define*

$$A_{r,j} = \frac{r+1}{p} + \frac{j}{q}, \quad j = 0, \dots, q-1,$$

$$B_{r,j} = \frac{r+1+j}{p}, \quad j = 0, \dots, p-1,$$

and let B_r be the list of $B_{r,j}$ with the single parameter equal to 1 omitted. Let

$$\lambda_{\varepsilon,\rho} = (-1)^p e^{\varepsilon \mathrm{i}\pi p\rho}\frac{q^q}{p^p},$$

$$C_{r,\varepsilon,\rho} = \frac{(-1)^r \varepsilon e^{\varepsilon \mathrm{i}\pi\rho(r+1)}}{2\pi\mathrm{i}\Gamma(r+1)}\Gamma\left(\frac{q(r+1)}{p}\right), \qquad \varepsilon = \pm 1.$$

Then

$$K_{p/q,\rho}(z) = \sum_{r=0}^{p-1}\sum_{\varepsilon=\pm 1} C_{r,\varepsilon,\rho} z^r {}_qF_{p-1}\left(\begin{matrix} A_{r,0}, \dots, A_{r,q-1} \\ B_r \end{matrix}; \lambda_{\varepsilon,\rho} z^p\right). \tag{II.18}$$

Proof. Split the sine in (II.14) into exponentials and collect indices $j = p\ell + r$. The ratio of successive coefficients in a fixed residue class is

$$(-1)^p e^{\varepsilon \mathrm{i}\pi p\rho} \frac{(q\ell + q(r+1)/p)_q}{(p\ell + r + 1)_p}.$$

Gauss multiplication converts this ratio to the hypergeometric ratio with numerator parameters $A_{r,j}$ and denominator parameters B_r, and the initial coefficient is $C_{r,\varepsilon,\rho}$. Summing over residue classes and exponential signs reconstructs the sine. ☐

Theorem II.8 (Rational residue representation, tail branch)**.** *Assume* $0 < \alpha = p/q < 1$ *with* $p < q$*. Put* $\sigma = e^{\mathrm{i}\pi p\rho/q}$*. For* $r = 1, \dots, q$*, define*

$$a_r = 1 + \frac{pr}{q}, \qquad A_{r,j} = \frac{a_r + j}{p}, \quad j = 0, \dots, p-1,$$

$$B_{r,j} = \frac{r + 1 + j}{q}, \quad j = 0, \dots, q-1,$$

$$\mu_{\varepsilon,\rho} = (-1)^q e^{\varepsilon \mathrm{i}\pi p\rho} \frac{p^p}{q^q},$$

$$D_{r,\varepsilon,\rho} = \frac{(p/q)(-1)^{r+1} \varepsilon e^{\varepsilon \mathrm{i}\pi p\rho r/q}}{2\pi \mathrm{i}\Gamma(r+1)} \Gamma\left(1 + \frac{pr}{q}\right).$$

Then

$$K_{p/q,\rho}(z) = \sum_{r=1}^{q} \sum_{\varepsilon=\pm 1} D_{r,\varepsilon,\rho} z^{-1-pr/q} {}_{p+1}F_q\left(\begin{matrix} 1, A_{r,0}, \dots, A_{r,p-1} \\ B_{r,0}, \dots, B_{r,q-1} \end{matrix}; \mu_{\varepsilon,\rho} z^{-p} \right). \tag{II.19}$$

Proof. Start from the exact tail series (II.15) and split the sine into exponentials. For a fixed sign $\varepsilon = \pm 1$ write the exponential coefficient of the k-th tail term as

$$c_k^{(\varepsilon)} = \frac{p/q}{\pi} \frac{(-1)^{k+1} e^{\varepsilon \mathrm{i}\pi p\rho k/q}}{\Gamma(k+1)} \Gamma\left(1 + \frac{pk}{q}\right).$$

Separate the index into congruence classes $k = q\ell + r$, $r = 1, \dots, q$. Then

$$\frac{c^{(\varepsilon)}_{q(\ell+1)+r}}{c^{(\varepsilon)}_{q\ell+r}} = (-1)^q e^{\varepsilon \mathrm{i}\pi p\rho} \frac{(1 + p\ell + pr/q)_p}{(q\ell + r + 1)_q}.$$

Gauss multiplication gives

$$\left(1 + p\ell + \frac{pr}{q}\right)_p = p^p \prod_{j=0}^{p-1} \left(\ell + \frac{1 + pr/q + j}{p}\right),$$

while

$$(q\ell + r + 1)_q = q^q \prod_{j=0}^{q-1} \left(\ell + \frac{r + 1 + j}{q}\right).$$

Therefore the coefficient ratio in the r-th residue class is the generalized hypergeometric ratio with numerator parameters $1, A_{r,0}, \dots, A_{r,p-1}$ and denominator parameters $B_{r,0}, \dots, B_{r,q-1}$, evaluated at $\mu_{\varepsilon,\rho} z^{-p}$. The numerator parameter 1 cancels the ordinary hypergeometric factorial $\ell!$, which is why the ratio above contains no extra $(\ell + 1)$ factor. The initial coefficient at $\ell = 0$ is exactly $D_{r,\varepsilon,\rho} z^{-1-pr/q}$. Summing over all classes $r = 1, \dots, q$ and over $\varepsilon = \pm 1$ reconstructs the sine and yields (II.19). ☐

Corollary II.9 (Holonomicity of rational half-line kernels)**.** *For fixed rational* $\alpha = p/q \in (0, 2] \setminus \{1\}$ *and* $0 < \rho < 1$*, the half-line kernel* $K_{\alpha,\rho}$ *is holonomic on its natural domain.*

Proof. Every generalized hypergeometric component in Theorems II.7 and II.8 is D-finite. D-finite functions are closed under algebraic changes of variables, multiplication by monomials, and finite sums. Hence the half-line kernel is holonomic. ☐

III The constructive rational-α varentropy formula

III-A Reduction to half-line kernel power moments

Define the half-line kernel power sum

$$R(\lambda) = \sum_{\rho \in \{\rho_+, \rho_-\}} I_\rho(\lambda), \qquad I_\rho(\lambda) = \int_0^\infty K_{\alpha,\rho}(z)^\lambda \, \mathrm{d}z. \tag{III.1}$$

By Lemma II.3, $R(1) = \alpha$.

Theorem III.1 (Scale and normalization cancellation)**.** *Let $Z_f(\lambda) = \int_{\mathbb{R}} f_{\alpha,b}(x)^\lambda \, \mathrm{d}x$. Then*

$$Z_f(\lambda) = A^{1-\lambda} \alpha^{-\lambda} R(\lambda),$$

and therefore

$$\mathrm{VE}_{\alpha,b} = \frac{R''(1)}{\alpha} - \left(\frac{R'(1)}{\alpha} \right)^2. \tag{III.2}$$

Proof. Using $x = sAz$ and (II.10),

$$f_{\alpha,b}(sAz)^\lambda A \, \mathrm{d}z = A^{1-\lambda} \alpha^{-\lambda} K_{\alpha,\rho_s}(z)^\lambda \, \mathrm{d}z.$$

Summing the two half-lines gives the displayed formula for $Z_f(\lambda)$. Taking logarithms,

$$\log Z_f(\lambda) = (1 - \lambda) \log A - \lambda \log \alpha + \log R(\lambda).$$

The first two terms are affine in λ, hence have zero second derivative. Using $R(1) = \alpha$ gives (III.2). ☐

III-B D-algebraicity of curvature integrands

Definition III.2 (D-algebraic function)**.** Let $U \subset \mathbb{C}$ be a domain and K a characteristic-zero coefficient field. A holomorphic function F on U is D-algebraic over $K(z)$ if there exists a nonzero differential polynomial

$$P \in K(z)[Y_0, Y_1, \dots, Y_m]$$

such that

$$P(F, F', \dots, F^{(m)}) = 0$$

on U.

Theorem III.3 (Logarithmic products are D-algebraic)**.** *Let H be a nonzero D-algebraic function on a simply connected domain U, and choose a branch of $\log H$ on U. Then*

$$H(\log H)^k$$

is D-algebraic on U for every fixed integer $k \geq 0$.

Proof. Since D-algebraic functions form a differential field and $H \neq 0$, the quotient H'/H is D-algebraic. If $L = \log H$, then $L' = H'/H$. D-algebraic functions are closed under antiderivatives: if g is D-algebraic and $L' = g$, an algebraic differential equation for g becomes one for L after replacing $g^{(j)}$ by $L^{(j+1)}$. Hence L is D-algebraic. Products and integer powers preserve D-algebraicity, so HL^k is D-algebraic. ☐

Corollary III.4 (Curvature integrands)**.** *For rational $\alpha = p/q \neq 1$ and $0 < \rho < 1$, the functions*

$$K_{\alpha,\rho}(z), \qquad K_{\alpha,\rho}(z) \log K_{\alpha,\rho}(z), \qquad K_{\alpha,\rho}(z)(\log K_{\alpha,\rho}(z))^2$$

are D-algebraic on every simply connected zero-free patch of $K_{\alpha,\rho}$.

Proof. By Corollary II.9, $K_{\alpha,\rho}$ is holonomic, hence D-finite and therefore D-algebraic. Apply Theorem III.3. ☐

Theorem III.5 (Hyperexponential exception)**.** *If a nonzero function H satisfies $H' = QH$ with $Q \in \mathbb{C}(z)$, then $H \log H$ and $H(\log H)^2$ are holonomic. In particular the residual logarithmic layer vanishes if one chooses $B = H$.*

Proof. Let $L = \log H$. Then $L' = Q$ and L is a finite sum of elementary logarithmic primitives of rational functions. The functions H, L, and L^2 are holonomic, and the class of holonomic functions is closed under products in one variable. Thus HL and HL^2 are holonomic. If $B = H$, then $N = H/B = 1$ and $\log N = 0$. ☐

III-C Endpoint-matched hyperexponential factor

Fix one half-line parameter ρ and write

$$H_\rho(z) = K_{\alpha,\rho}(z).$$

Set

$$H_{0,\rho} = \frac{\Gamma(1/\alpha)}{\pi}\sin(\pi\rho), \qquad c_{\infty,\rho} = \frac{\alpha}{\pi}\Gamma(1+\alpha)\sin(\pi\alpha\rho). \tag{III.3}$$

Lemma III.6 (Endpoint constants of the half-line kernel)**.** *Let ρ be one of the two half-line parameters associated with a strict non-extremal stable law, so that $0 < \rho < 1$ and $0 < \alpha\rho < 1$. Then the constants in (III.3) are positive and satisfy*

$$K_{\alpha,\rho}(z) = H_{0,\rho} + o(1), \qquad z \downarrow 0, \tag{III.4}$$

$$K_{\alpha,\rho}(z) \sim c_{\infty,\rho} z^{-1-\alpha}, \qquad z \to \infty. \tag{III.5}$$

Proof. The positivity follows from $0 < \rho < 1$ and, in the strict non-extremal range, $0 < \alpha\rho < 1$. To identify the constants, use the Mellin–Barnes formula (II.11). As $z \downarrow 0$, shift the contour left across the pole $r = 0$ of $\Gamma(r)$. Its residue is one, and the remaining factors give

$$\frac{1}{\pi}\Gamma(1/\alpha)\sin(\pi\rho) = H_{0,\rho}.$$

The shifted contour contributes $o(1)$ by the standard Mellin–Barnes asymptotic estimate obtained after moving the contour a fixed distance to the left, giving (III.4). As $z \to \infty$, shift the contour right. The first nonzero pole of $\Gamma((1-r)/\alpha)$ is at $r = 1+\alpha$; the pole at $r = 1$ is killed by the sine factor. The standard residue calculation gives the leading term

$$\frac{\alpha}{\pi}\Gamma(1+\alpha)\sin(\pi\alpha\rho)z^{-1-\alpha},$$

which is (III.5); the remainder is lower order after shifting the contour farther to the right. These are the leading terms of the Hatzinikitas–Pachos expansions quoted in Theorems II.4 and II.5 in the regimes where those series are convergent, and are asymptotic leading terms in the complementary regimes. ☐

Let

$$\nu = \frac{1+\alpha}{p} = \frac{p+q}{pq}, \qquad \kappa_\rho = \left(\frac{H_{0,\rho}}{c_{\infty,\rho}}\right)^{1/\nu}, \tag{III.6}$$

where the positive real root is chosen.

Definition III.7 (Endpoint-matched hyperexponential factor)**.** The endpoint-matched hyperexponential factor for the half-line parameter ρ is

$$B_\rho(z) = H_{0,\rho}(1 + \kappa_\rho z^p)^{-\nu}. \tag{III.7}$$

The normalized residual is

$$N_\rho(z) = \frac{H_\rho(z)}{B_\rho(z)}. \tag{III.8}$$

Theorem III.8 (Endpoint normalization and hyperexponentiality)**.** *The endpoint-matched hyperexponential factor satisfies*

$$B_\rho(0+) = H_{0,\rho}, \qquad B_\rho(z) \sim c_{\infty,\rho} z^{-1-\alpha} \quad (z \to \infty). \tag{III.9}$$

Consequently

$$N_\rho(0+) = 1, \qquad N_\rho(\infty) = 1. \tag{III.10}$$

Moreover, with the Euler, or dilation, derivative $\delta = z\,\mathrm{d}/\mathrm{d}z$,

$$r_{B,\rho}(z) := \frac{\delta B_\rho}{B_\rho} = -(1+\alpha)\frac{\kappa_\rho z^p}{1+\kappa_\rho z^p}. \tag{III.11}$$

Thus B_ρ is hyperexponential. If H_ρ is holonomic, then $N_\rho = H_\rho/B_\rho$ is holonomic.

Proof. At $z = 0$, (III.7) gives $B_\rho(0+) = H_{0,\rho}$. At infinity,

$$B_\rho(z) \sim H_{0,\rho}\kappa_\rho^{-\nu} z^{-p\nu}.$$

Since $p\nu = 1 + \alpha$ and $\kappa_\rho^\nu = H_{0,\rho}/c_{\infty,\rho}$, the leading constant is $c_{\infty,\rho}$. The endpoint limits for N_ρ follow by comparing these two limits with the kernel asymptotics in Lemma III.6.

Differentiating (III.7) by the Euler/dilation derivative gives (III.11). Hence B_ρ has rational logarithmic derivative. If $L(z, \delta)H_\rho = 0$ is a holonomic equation for H_ρ, then $H_\rho = B_\rho N_\rho$ implies

$$\delta(B_\rho N_\rho) = B_\rho(\delta + r_{B,\rho})N_\rho.$$

Thus N_ρ is annihilated by the gauge-conjugated operator $L(z, \delta + r_{B,\rho})$. ☐

III-D Exact factor moments

Define

$$F_\rho(t; \kappa_\rho) = \int_0^\infty H_\rho(z)(1 + \kappa_\rho z^p)^{-t}\, \mathrm{d}z. \tag{III.12}$$

Then $F_\rho(0; \kappa_\rho) = \alpha\rho$.

Theorem III.9 (Exact factor moments)**.** *Let*

$$\mathcal{B}_{1,\rho} := \int_0^\infty H_\rho(z) \log B_\rho(z)\, \mathrm{d}z, \qquad \mathcal{B}_{2,\rho} := \int_0^\infty H_\rho(z)(\log B_\rho(z))^2\, \mathrm{d}z.$$

Then

$$\mathcal{B}_{1,\rho} = \alpha\rho \log H_{0,\rho} + \nu F_\rho'(0; \kappa_\rho), \tag{III.13}$$

and

$$\mathcal{B}_{2,\rho} = \alpha\rho(\log H_{0,\rho})^2 + 2\nu(\log H_{0,\rho})F_\rho'(0; \kappa_\rho) + \nu^2 F_\rho''(0; \kappa_\rho). \tag{III.14}$$

Proof. From (III.7),

$$\log B_\rho(z) = \log H_{0,\rho} - \nu \log(1 + \kappa_\rho z^p).$$

Also

$$F_\rho'(0; \kappa_\rho) = -\int_0^\infty H_\rho(z) \log(1 + \kappa_\rho z^p)\, \mathrm{d}z,$$

and

$$F_\rho''(0; \kappa_\rho) = \int_0^\infty H_\rho(z) \log^2(1 + \kappa_\rho z^p)\, \mathrm{d}z.$$

For the first moment,

$$\begin{aligned}\mathcal{B}_{1,\rho} &= (\log H_{0,\rho})\int_0^\infty H_\rho(z)\, \mathrm{d}z - \nu \int_0^\infty H_\rho(z) \log(1 + \kappa_\rho z^p)\, \mathrm{d}z \\ &= \alpha\rho \log H_{0,\rho} + \nu F_\rho'(0; \kappa_\rho),\end{aligned}$$

which proves (III.13). For the second moment, square the displayed expression for $\log B_\rho$:

$$(\log B_\rho)^2 = (\log H_{0,\rho})^2 - 2\nu(\log H_{0,\rho}) \log(1 + \kappa_\rho z^p) + \nu^2 \log^2(1 + \kappa_\rho z^p).$$

Integrating term by term, using $\int H_\rho = \alpha\rho$, and substituting the formulas for $F_\rho'(0)$ and $F_\rho''(0)$ gives (III.14). ☐

The derivatives $F_\rho'(0)$ and $F_\rho''(0)$ are evaluated by residue formulas. The formulas are built from standard primitives: ordinary generalized hypergeometric functions and explicit Pochhammer–polygamma sums. We give the central-branch formula in the main text and place the tail formula in Appendix B.

Theorem III.10 (Central right-residue factor derivatives)**.** *Assume $1 < \alpha = p/q < 2$, so $p > q$. Let*

$$L = \log \kappa_\rho, \qquad P = \pi p\rho.$$

We write $H_N = \sum_{j=1}^N j^{-1}$ and $H_N^{(2)} = \sum_{j=1}^N j^{-2}$ for the ordinary and second-order harmonic numbers, and ψ_1 for the trigamma function. Then

$$F_\rho'(0; \kappa_\rho) = F_{0,\rho}^{(1)} + F_{\mathrm{ni},\rho}^{(1)} + F_{\mathrm{int},\rho}^{(1)}, \tag{III.15}$$

$$F''_\rho(0;\kappa_\rho) = F^{(2)}_{0,\rho} + F^{(2)}_{\mathrm{ni},\rho} + F^{(2)}_{\mathrm{int},\rho}. \tag{III.16}$$

The $\upsilon = 0$ terms are

$$F^{(1)}_{0,\rho} = \alpha\rho\big(\gamma_E(p-q) - L\big), \tag{III.17}$$

$$F^{(2)}_{0,\rho} = \alpha\rho\left[\big(L + \gamma_E(q-p)\big)^2 + \frac{\pi^2}{6}\big(p^2 + q^2 - 2p^2\rho^2\big)\right]. \tag{III.18}$$

For $r = 1, \ldots, p-1$, let

$$a_r = \frac{r}{p}, \qquad \mathbf{A}_r = \left(a_r, a_r + \frac{1}{q}, \ldots, a_r + \frac{q-1}{q}\right),$$

$$\mathbf{B}_r = \left(\frac{r+1}{p}, \frac{r+2}{p}, \ldots, \frac{r+p}{p}\right) \setminus \{1\},$$

$$\zeta_{\rho,\pm} = (-1)^{p-1}\frac{q^q}{p^p\kappa_\rho}e^{\pm \mathrm{i}\pi p\rho},$$

$$C_{\rho,r} = \frac{(-1)^r\kappa_\rho^{-r/p}\Gamma(qr/p)}{\Gamma(r+1)\sin(\pi r/p)}.$$

Then

$$\begin{aligned} F^{(1)}_{\mathrm{ni},\rho} = \sum_{r=1}^{p-1}\frac{C_{\rho,r}}{2\mathrm{i}}\Bigg[&e^{\mathrm{i}\pi\rho r}\,{}_qF_{p-1}\left(\begin{matrix}\mathbf{A}_r\\ \mathbf{B}_r\end{matrix};\zeta_{\rho,+}\right)\\ &- e^{-\mathrm{i}\pi\rho r}\,{}_qF_{p-1}\left(\begin{matrix}\mathbf{A}_r\\ \mathbf{B}_r\end{matrix};\zeta_{\rho,-}\right)\Bigg]. \end{aligned} \tag{III.19}$$

The second noninteger part is

$$F^{(2)}_{\mathrm{ni},\rho} = \sum_{r=1}^{p-1}\frac{C_{\rho,r}}{\mathrm{i}}\left[e^{\mathrm{i}\pi\rho r}\mathcal{L}_{\rho,r}(\zeta_{\rho,+}) - e^{-\mathrm{i}\pi\rho r}\mathcal{L}_{\rho,r}(\zeta_{\rho,-})\right], \tag{III.20}$$

where the Pochhammer–polygamma sum is

$$\mathcal{L}_{\rho,r}(\zeta) = \sum_{n=0}^{\infty}\frac{\prod_{a\in\mathbf{A}_r}(a)_n}{\prod_{b\in\mathbf{B}_r}(b)_n\,n!}\zeta^n\left[\psi\left(n+1+\frac{r}{p}\right) + \gamma_E + \pi\cot\left(\frac{\pi r}{p}\right)\right]. \tag{III.21}$$

Finally, define for $N \geq 1$

$$S_N = \sin(\pi p\rho N), \qquad C_N = \cos(\pi p\rho N),$$

$$\ell_N = q\psi(qN) - L - \frac{1}{N}, \qquad b_N = q^2\psi_1(qN) + \frac{1}{N^2},$$

$$G_N = \frac{\Gamma(qN)\kappa_\rho^{-N}}{N},$$

$$h^{(p)}_N = \psi(pN), \qquad c^{(p)}_N = \frac{1}{2}\left(\psi(pN)^2 + 2\zeta(2) - \psi_1(pN)\right),$$

$$A_N = \frac{(-1)^{pN-1}}{\Gamma(pN)}, \quad a_{-1,N} = -\frac{A_N}{p}, \quad a_{0,N} = A_N h^{(p)}_N, \quad a_{1,N} = -A_N p\, c^{(p)}_N.$$

Set

$$g_{0,N} = G_N S_N,$$

$$g_{1,N} = G_N(\ell_N S_N + PC_N),$$

$$g_{2,N} = \frac{G_N}{2}\left[(\ell_N^2 + b_N - P^2)S_N + 2\ell_N PC_N\right].$$

Then

$$d_{-2,N} = \frac{(-1)^N}{\pi}a_{-1,N}g_{0,N},$$

$$d_{-1,N} = \frac{(-1)^N}{\pi}(a_{-1,N}g_{1,N} + a_{0,N}g_{0,N}),$$

$$d_{0,N} = \frac{(-1)^N}{\pi}(a_{-1,N}g_{2,N} + a_{0,N}g_{1,N} + a_{1,N}g_{0,N}) + (-1)^N\frac{\pi}{6}a_{-1,N}g_{0,N}.$$

The integer collision parts are

$$F^{(1)}_{\mathrm{int},\rho} = \sum_{N=1}^{\infty} d_{-1,N}, \tag{III.22}$$

$$F^{(2)}_{\mathrm{int},\rho} = 2\sum_{N=1}^{\infty}\left[d_{0,N} + H_N d_{-1,N} - (\zeta(2) + H_N^{(2)})d_{-2,N}\right]. \tag{III.23}$$

All infinite sums in (III.21)*,* (III.22)*, and* (III.23) *are convergent for fixed* $\kappa_\rho \neq 0$*.*

Proof. The Mellin transform of $\log(1+x)$ and $\log^2(1+x)$ gives

$$F'_\rho(0;\kappa_\rho) = -\frac{1}{2\pi \mathrm{i}}\int_C \frac{\Gamma(1-pv)\Gamma(qv)\sin(\pi p\rho v)\kappa_\rho^{-v}}{v\sin(\pi v)}\,\mathrm{d}v,$$

$$F''_\rho(0;\kappa_\rho) = -\frac{2}{2\pi \mathrm{i}}\int_C \frac{\Gamma(1-pv)\Gamma(qv)\sin(\pi p\rho v)\kappa_\rho^{-v}}{v\sin(\pi v)}(\psi(-v)+\gamma_E)\,\mathrm{d}v,$$

with $-1/q < \Re v < 0$. Since $p > q$, closing to the right produces convergent residue classes. The pole at $v = 0$ gives (III.17) and (III.18); equivalently these terms are the Mellin-log moments of the monomial approximation $\log\kappa_\rho + p\log z$.

The noninteger poles $v = n + r/p$ come from $\Gamma(1-pv)$. Taking residues, splitting $\sin(\pi p\rho v)$ into exponentials, and using Gauss multiplication gives the hypergeometric family (III.19). In the second derivative, the extra multiplier $\psi(-v)+\gamma_E$ is reflected to

$$\psi\left(n+1+\frac{r}{p}\right) + \gamma_E + \pi\cot\left(\frac{\pi r}{p}\right),$$

which gives the explicit Pochhammer–polygamma sum (III.21) and hence (III.20).

At integer poles $v = N \geq 1$, $\Gamma(1-pv)$ and $1/\sin(\pi v)$ collide. The full Laurent bookkeeping is recorded in Lemma C.1. With $w = v - N$, that lemma gives

$$\frac{\Gamma(1-pv)\Gamma(qv)\sin(\pi p\rho v)\kappa_\rho^{-v}}{v\sin(\pi v)} = d_{-2,N}w^{-2} + d_{-1,N}w^{-1} + d_{0,N} + O(w).$$

Thus the residue of the first derivative kernel is $d_{-1,N}$, giving (III.22). For the second derivative, Lemma C.1 also gives

$$\psi(-N-w) + \gamma_E = \frac{1}{w} + H_N - (\zeta(2) + H_N^{(2)})w + O(w^2),$$

so the coefficient of w^{-1} in the product is

$$d_{0,N} + H_N d_{-1,N} - (\zeta(2) + H_N^{(2)})d_{-2,N}.$$

This gives (III.23). Finally, because $p > q$, the integer terms contain $\Gamma(qN)/\Gamma(pN)$ and thus decay superexponentially by Stirling's formula. The noninteger terms are of type ${}_qF_{p-1}$ and logarithmic variants with the same ratio, so they converge as well. ☐

III-E Residual moments and the canonical logit atlas

The factor moments are exact. The remaining terms are the moments of the normalized residual

$$N_\rho(z) = H_\rho(z)/B_\rho(z).$$

Define

$$\mathcal{R}_{1,\rho} := \int_0^\infty H_\rho(z)\log N_\rho(z)\,\mathrm{d}z, \tag{III.24}$$

$$\mathcal{R}_{2,\rho} := \int_0^\infty H_\rho(z)\left[(\log H_\rho(z))^2 - (\log B_\rho(z))^2\right]\,\mathrm{d}z. \tag{III.25}$$

Equivalently,

$$\mathcal{R}_{2,\rho} = \int_0^\infty \left[2(\log B_\rho)H_\rho\log N_\rho + H_\rho(\log N_\rho)^2\right]\,\mathrm{d}z. \tag{III.26}$$

The residual D-algebraic equations are recorded in Appendix D. For the actual constructive formula it is better to use the endpoint-matched factor itself to compactify the half-line. This removes the arbitrary origin/body/infinity split that appears in a naive patch implementation.

Theorem III.11 (Local power-coefficient recurrence)**.** *Let*

$$A(u) = \sum_{n\geq 0} a_n u^n, \qquad a_0 > 0,$$

be analytic and nonzero on a disk. Choose the branch of $A(u)^\lambda$ determined by the positive value a_0^λ, and write

$$A(u)^\lambda = \sum_{n\geq 0} D_n(\lambda) u^n.$$

Then

$$D_0(\lambda) = a_0^\lambda,$$

and, for $n \geq 1$,

$$D_n(\lambda) = \frac{1}{na_0} \sum_{j=1}^{n} ((\lambda+1)j - n) a_j D_{n-j}(\lambda). \tag{III.27}$$

Here $[u^n]F(u)$ denotes the coefficient of u^n in the Taylor expansion of F at $u = 0$. With

$$P_n = D_n'(1) = [u^n] A \log A, \qquad Q_n = D_n''(1) = [u^n] A (\log A)^2,$$

one has

$$P_0 = a_0 \log a_0, \qquad P_n = \frac{1}{na_0} \sum_{j=1}^{n} a_j \big(j a_{n-j} + (2j - n) P_{n-j} \big), \tag{III.28}$$

$$Q_0 = a_0 (\log a_0)^2, \qquad Q_n = \frac{1}{na_0} \sum_{j=1}^{n} a_j \big(2j P_{n-j} + (2j - n) Q_{n-j} \big). \tag{III.29}$$

Proof. Let $C(u, \lambda) = A(u)^\lambda$. The identity

$$A \partial_u C = \lambda A' C$$

implies, after equating the coefficient of u^{n-1},

$$a_0 n D_n + \sum_{j=1}^{n-1} a_j (n - j) D_{n-j} = \lambda \sum_{j=1}^{n} j a_j D_{n-j}.$$

Solving for D_n gives (III.27). Differentiating once at $\lambda = 1$, using $D_n(1) = a_n$, gives (III.28). Differentiating a second time gives two copies of the cross term $j D_{n-j}'(1)$ and produces (III.29). ☐

Define the compactifying coordinate

$$y = \frac{\kappa_\rho z^p}{1 + \kappa_\rho z^p}, \qquad z = \kappa_\rho^{-1/p} \left(\frac{y}{1 - y} \right)^{1/p}, \qquad 0 < y < 1. \tag{III.30}$$

All fractional powers in this compactification are taken on the positive real branch on $(0, 1)$; on each complex cell neighborhood we use the analytic continuation of that branch. Then

$$B_\rho(y) = H_{0,\rho} (1 - y)^\nu, \qquad \Phi_\rho(y) := \log B_\rho(y) = \log H_{0,\rho} + \nu \log(1 - y). \tag{III.31}$$

Put

$$N_\rho(y) = \frac{H_\rho(z(y))}{H_{0,\rho}(1 - y)^\nu}. \tag{III.32}$$

By Theorem III.8,

$$N_\rho(0) = 1, \qquad N_\rho(1) = 1.$$

The Jacobian gives the exact one-interval identities

$$H_\rho(z)\, \mathrm{d}z = C_\rho\, y^{1/p-1} (1 - y)^{1/q-1} N_\rho(y)\, \mathrm{d}y, \qquad C_\rho = \frac{H_{0,\rho}}{p \kappa_\rho^{1/p}}. \tag{III.33}$$

Therefore

$$\mathcal{R}_{1,\rho} = C_\rho \int_0^1 y^{1/p-1} (1 - y)^{1/q-1} N_\rho(y) \log N_\rho(y)\, \mathrm{d}y, \tag{III.34}$$

$$\mathcal{R}_{2,\rho} = C_\rho \int_0^1 y^{1/p-1}(1-y)^{1/q-1} N_\rho(y) \times \left[2\Phi_\rho(y)\log N_\rho(y) + (\log N_\rho(y))^2\right]\,\mathrm{d}y. \tag{III.35}$$

For coefficient evaluation, pass to the logit coordinate

$$t = \log\frac{y}{1-y} = \log(\kappa_\rho z^p), \qquad y = \frac{e^t}{1+e^t}, \qquad -\infty < t < \infty. \tag{III.36}$$

Then

$$W_\rho(t) = C_\rho e^{t/p}(1+e^t)^{-\nu}, \qquad \Phi_\rho(t) = \log H_{0,\rho} - \nu\log(1+e^t), \tag{III.37}$$

and

$$\mathcal{R}_{1,\rho} = \int_{-\infty}^{\infty} W_\rho(t)N_\rho(t)\log N_\rho(t)\,\mathrm{d}t, \tag{III.38}$$

$$\mathcal{R}_{2,\rho} = \int_{-\infty}^{\infty} W_\rho(t)\left[2\Phi_\rho(t)N_\rho(t)\log N_\rho(t) + N_\rho(t)(\log N_\rho(t))^2\right]\,\mathrm{d}t. \tag{III.39}$$

Here $N_\rho(t)$ means $N_\rho(y(t))$.

Define the two residual logit integrands

$$G_{1,\rho}(t) = W_\rho(t)N_\rho(t)\log N_\rho(t), \tag{III.40}$$

$$G_{2,\rho}(t) = W_\rho(t)\left[2\Phi_\rho(t)N_\rho(t)\log N_\rho(t) + N_\rho(t)(\log N_\rho(t))^2\right]. \tag{III.41}$$

We now prove that the zero-free residual atlas needed for coefficient evaluation exists. The proof has two parts. First, endpoint matching gives uniform exponential decay on both logit tails. Second, positivity of the stable density on the real axis excludes real zeros; discreteness of complex zeros then gives a finite dyadic refinement on the remaining compact block.

An *atlas cell* is a compact interval

$$J = [c_J - h_J, c_J + h_J], \qquad h_J > 0,$$

with real center c_J. It is called Taylor-admissible if there is a radius $r_J > h_J$ such that N_ρ, W_ρ, and Φ_ρ are holomorphic on the disk

$$D_J = \{t \in \mathbb{C} : |t - c_J| < r_J\},$$

and N_ρ has no zero in D_J. On such a disk the real-positive branch of $\log N_\rho$ on J extends uniquely and holomorphically.

Lemma III.12 (Sectorial endpoint estimates for the residual)**.** *Let $\alpha = p/q \in (0,2)\setminus\{1\}$ be rational, and let $0 < \rho < 1$ be strict non-extremal, so $0 < \alpha\rho < 1$. In the logit coordinate $t = \log(\kappa_\rho z^p)$, there is a number η with $1/2 < \eta < \pi$ such that, uniformly for $|\Im t| \le \eta$,*

$$N_\rho(t) - 1 = O(e^{t/p}) \qquad (\Re t \to -\infty), \tag{III.42}$$

$$N_\rho(t) - 1 = O(e^{-t/q}) \qquad (\Re t \to +\infty). \tag{III.43}$$

Consequently, after increasing T if necessary, N_ρ is zero-free in the two tail strips

$$\{t : \Re t < -T,\ |\Im t| \le \eta\}, \qquad \{t : \Re t > T,\ |\Im t| \le \eta\},$$

and the logarithm in (III.40)–(III.41) *is the branch satisfying* $\log N_\rho(t) \to 0$ *at both ends.*

Proof. Put $z(t) = \kappa_\rho^{-1/p}e^{t/p}$. For finite t, the rational residue representations of Theorems II.7 and II.8 show that $H_\rho(z(t))$ is holomorphic in t: the powers of z become exponentials in t, and the hypergeometric arguments become constant multiples of e^t in the central case and of e^{-t} in the tail case. The only singularities of the explicit factor $B_\rho(t) = H_{0,\rho}(1+e^t)^{-\nu}$ and of $\Phi_\rho(t)$ in the logarithmic coordinate occur at $1 + e^t = 0$, so the strip $|\Im t| < \pi$ is available after fixing the principal branch of $\log(1+e^t)$ there.

The endpoint estimates for H_ρ are the sectorial form of Lemma III.6. More explicitly, shifting the Mellin–Barnes contour in (II.11) left past the poles $r = 0$ and $r = -1$, and keeping the $r = -1$ residue inside the error term, gives

$$H_\rho(z) = H_{0,\rho}\big(1 + O(z)\big) \qquad (z \to 0)$$

uniformly in any sufficiently narrow sector about the positive real axis. Shifting the contour to the right past the tail poles $r = 1 + \alpha$ and $r = 1 + 2\alpha$, and keeping the second tail residue inside the relative error term, gives

$$H_\rho(z) = c_{\infty,\rho}z^{-1-\alpha}\big(1 + O(z^{-\alpha})\big) \qquad (z \to \infty)$$

uniformly in such sectors. In the regimes where the central or tail residue series is convergent, these are also immediate from Theorems II.4 and II.5; the Mellin–Barnes estimate supplies the complementary endpoint.

Choose η with $1/2 < \eta < \pi$ small enough that the sector traced by $z(t)$ for $|\Im t| \le \eta$ is one of the sectors above. As $\Re t \to -\infty$,

$$B_\rho(t) = H_{0,\rho}(1+e^t)^{-\nu} = H_{0,\rho}\big(1 + O(e^t)\big),$$

whereas $z(t) = O(e^{t/p})$. Since $p \ge 1$, this gives

$$\frac{H_\rho(z(t))}{B_\rho(t)} - 1 = O(e^{t/p}),$$

which proves (III.42). As $\Re t \to +\infty$,

$$B_\rho(t) = H_{0,\rho}e^{-\nu t}\big(1 + O(e^{-t})\big).$$

Because $p\nu = 1+\alpha$ and $\kappa_\rho^\nu = H_{0,\rho}/c_{\infty,\rho}$, this is the same as

$$B_\rho(t) = c_{\infty,\rho}z(t)^{-1-\alpha}\big(1 + O(e^{-t})\big).$$

The kernel estimate above then gives

$$\frac{H_\rho(z(t))}{B_\rho(t)} - 1 = O(e^{-t/q}) + O(e^{-t}) = O(e^{-t/q}),$$

which proves (III.43). The two estimates imply $|N_\rho(t) - 1| < 1/2$ in both tail strips after T is chosen large enough. Hence N_ρ is zero-free there and the branch $\log N_\rho = \log(1 + (N_\rho - 1))$ is single-valued and tends to zero. ☐

Lemma III.13 (Tail decay of the residual integrands)**.** *Under the hypotheses of Lemma III.12, there are constants $C, c, T > 0$ such that, for $|\Im t| \le \eta$ and $|\Re t| > T$,*

$$|G_{1,\rho}(t)| + |G_{2,\rho}(t)| \le C(1+|t|)e^{-c|\Re t|}. \tag{III.44}$$

Consequently the residual integrals (III.38) *and* (III.39) *are absolutely convergent.*

Proof. In the strip $|\Im t| \le \eta$, Lemma III.12 gives $\log N_\rho(t) = O(e^{t/p})$ as $\Re t \to -\infty$ and $\log N_\rho(t) = O(e^{-t/q})$ as $\Re t \to +\infty$. Also

$$W_\rho(t) = C_\rho e^{t/p}(1+e^t)^{-\nu} = O(e^{t/p}) \qquad (\Re t \to -\infty),$$

and, since $\nu = 1/p + 1/q$,

$$W_\rho(t) = O(e^{-t/q}) \qquad (\Re t \to +\infty).$$

Finally

$$\Phi_\rho(t) = \log H_{0,\rho} - \nu \log(1+e^t)$$

is bounded as $\Re t \to -\infty$ and is $O(1+|t|)$ as $\Re t \to +\infty$. Substituting these estimates in (III.40)–(III.41) gives the stated bound, and the bound is integrable on both tails. ☐

Lemma III.14 (Existence and summability of a zero-free residual atlas)**.** *For each strict non-extremal rational half-line kernel H_ρ, there exists a countable Taylor-admissible dyadic logit atlas $\mathfrak{A}_\rho$ with disjoint interiors and union $\mathbb{R}$. The atlas may be chosen so that all but finitely many of its cells are the unit cells $[\ell - 1/2, \ell + 1/2]$. For this atlas, the Taylor series defining the coefficients below converge absolutely on every cell, and their absolute Taylor majorants are summable over $\mathfrak{A}_\rho$ for both residual integrands.*

Proof. Choose $\eta > 1/2$ and T as in Lemmas III.12 and III.13. Let r satisfy $1/2 < r < \eta$. For all sufficiently large positive or negative integers ℓ, the disk $|t - \ell| < r$ lies in one of the two zero-free tail strips. Hence all sufficiently far-out unit cells are Taylor-admissible.

It remains to cover a compact real interval K. On the strip $|\Im t| < \pi$, the functions W_ρ and Φ_ρ are holomorphic after the branch of $\log(1+e^t)$ is fixed, and N_ρ is holomorphic. The stable density is strictly positive on the real line in the non-extremal case, and $B_\rho(t) > 0$ for real t, so $N_\rho(t) > 0$ on K. For each $x \in K$, continuity gives a disk $D(x, r_x)$ on which N_ρ has no zero and on which W_ρ and Φ_ρ are holomorphic. A finite subcover of these disks covers K.

By the Lebesgue-number lemma, after subdividing the finitely many unit cells meeting K into sufficiently fine dyadic subcells, each subcell $J = [c_J - h_J, c_J + h_J]$ has the property that the larger disk $|t - c_J| < 2h_J$ is contained in one member of the finite zero-free cover. Thus each such J is Taylor-admissible, with $r_J = 2h_J$.

For the summability statement, use Cauchy's estimate. On a tail unit cell centered at ℓ, take the disk $|t-\ell| < r$ with $r > 1/2$. Lemma III.13 bounds the suprema of $G_{1,\rho}$ and $G_{2,\rho}$ on these disks by

$$M_\ell \le C(1+|\ell|)e^{-c|\ell|}.$$

If $G(\ell+u) = \sum g_{\ell,n}u^n$ on a unit tail cell, then $|g_{\ell,n}| \le M_\ell r^{-n}$. Since

$$\sum_{n\ge 0} r^{-n}\int_{-1/2}^{1/2} |u|^n \,\mathrm{d}u < \infty \qquad (r > 1/2),$$

the absolute Taylor contribution of that cell is at most a constant times M_ℓ. The sum of the M_ℓ over the two tails is finite. The compact part has only finitely many subcells, and each has a convergence radius strictly larger than its half-length. Its absolute majorants therefore contribute a finite amount. This proves the claimed summability and the absolute convergence of the residual integrals. ☐

On a cell $J \in \mathfrak{A}_\rho$, write $t = c_J + u$, $-h_J \le u \le h_J$, and expand

$$N_\rho(c_J+u) = \sum_{n\ge 0} a_{J,n}u^n, \quad W_\rho(c_J+u) = \sum_{n\ge 0} w_{J,n}u^n, \quad \Phi_\rho(c_J+u) = \sum_{n\ge 0} \phi_{J,n}u^n. \tag{III.45}$$

The coefficients $a_{J,n}$ are constructive: expand the holonomic half-line kernel $H_\rho(z(c_J+u))$ and divide by the explicit t-coordinate factor $B_\rho(c_J+u) = H_{0,\rho}(1+e^{c_J+u})^{-\nu}$, or equivalently multiply by its reciprocal Taylor series. The coefficients $w_{J,n}$ and $\phi_{J,n}$ come from the explicit functions in (III.37). Since $N_\rho(c_J) > 0$ on the real axis, the constant term $a_{J,0}$ is positive. Apply Theorem III.11 to the series $N_\rho(c_J+u)$ and write

$$P_{J,n} = [u^n]N_\rho(c_J+u)\log N_\rho(c_J+u), \qquad Q_{J,n} = [u^n]N_\rho(c_J+u)(\log N_\rho(c_J+u))^2. \tag{III.46}$$

Define the cell residual coefficients

$$C^{(1)}_{\rho,J,n} = \sum_{i=0}^{n} w_{J,i}P_{J,n-i}, \tag{III.47}$$

$$C^{(2)}_{\rho,J,n} = \sum_{i=0}^{n} w_{J,i}\left(Q_{J,n-i} + 2\sum_{j=0}^{n-i}\phi_{J,j}P_{J,n-i-j}\right). \tag{III.48}$$

Finally let

$$\Omega_{J,n} = \int_{-h_J}^{h_J} u^n \,\mathrm{d}u = \begin{cases} 0, & n \text{ odd},\\ \dfrac{2h_J^{n+1}}{n+1}, & n \text{ even}.\end{cases} \tag{III.49}$$

For the unrefined unit cell $I_\ell = [\ell-1/2, \ell+1/2]$, this reduces to $\Omega_{I_\ell,n} = 0$ for odd n and $1/(2^n(n+1))$ for even n.

Theorem III.15 (Zero-free atlas residual formula)**.** *Let $\mathfrak{A}_\rho$ be a Taylor-admissible dyadic atlas supplied by Lemma III.14. Then*

$$\mathcal{R}_{1,\rho} = \sum_{J\in\mathfrak{A}_\rho}\sum_{n\ge 0} C^{(1)}_{\rho,J,n}\Omega_{J,n}, \tag{III.50}$$

$$\mathcal{R}_{2,\rho} = \sum_{J\in\mathfrak{A}_\rho}\sum_{n\ge 0} C^{(2)}_{\rho,J,n}\Omega_{J,n}. \tag{III.51}$$

Both series converge absolutely.

Proof. Equations (III.38) and (III.39) express the residual moments as integrals over the real t-line. On a cell J, substitute $t = c_J + u$. The recurrence of Theorem III.11 applied to $N_\rho(c_J+u)$ gives the coefficients $P_{J,n}$ and $Q_{J,n}$ in (III.46). Multiplying by the Taylor series for W_ρ and for Φ_ρ gives the convolution formulas (III.47) and (III.48). Since each cell is Taylor-admissible, these Taylor products converge absolutely on the closed integration interval. Termwise integration gives the moments (III.49). Lemma III.14 supplies summable absolute Taylor majorants over all cells, so Tonelli's theorem permits summation over $J \in \mathfrak{A}_\rho$ and proves (III.50)–(III.51). ☐

Remark III.16 (Only finitely many refinements)*.* The unit logit cells remain the canonical presentation of the atlas. Lemma III.14 shows that the two tails are already zero-free after a finite threshold, because $N_\rho(t) \to 1$ uniformly in a complex strip. Thus dyadic refinement is needed only in a compact block, where positivity on the real axis and discreteness of complex zeros guarantee termination.

Remark III.17 (Endpoint normalization and tail decay). The coordinate y incorporates both endpoint matching conditions. As $t \to -\infty$, one has $y \to 0$ and $N_\rho(t) = 1 + O(e^{t/p})$; as $t \to +\infty$, one has $y \to 1$ and $N_\rho(t) = 1 + O(e^{-t/q})$. Moreover $W_\rho(t) \sim C_\rho e^{t/p}$ as $t \to -\infty$ and $W_\rho(t) \sim C_\rho e^{-t/q}$ as $t \to +\infty$. The second residual has the additional factor $\Phi_\rho(t) \sim -\nu t$, but this linear factor is still exponentially integrable. Thus a numerical evaluator may truncate the logit-cell sum by a tolerance as opposed to by arbitrary endpoint cutoffs.

III-F Constructive certificate for residual periods

The word "constructive" in Theorem III.21 means more than formal closure under D-algebraic operations. For each half-line parameter ρ and for each requested tolerance, the residual computation is represented by finite certificate data whose validity can be checked independently.

Definition III.18 (Residual evaluation certificate). Fix $k \in \{1, 2\}$, where $G_{1,\rho}$ and $G_{2,\rho}$ are the logit residual integrands in (III.40)–(III.41). A residual evaluation certificate for $\mathcal{R}_{k,\rho}$ consists of the following finite data.

(i) A tail threshold L and a finite dyadic subatlas $\mathfrak{A}_{\rho,L} \subset \mathfrak{A}_\rho$ covering $[-L, L]$, together with the assertion that outside $[-L, L]$ the atlas consists of unrefined unit cells.

(ii) For each cell $J = [c_J - h_J, c_J + h_J] \in \mathfrak{A}_{\rho,L}$, a disk radius $r_J > h_J$ on which N_ρ, W_ρ, and Φ_ρ are analytic and N_ρ is zero-free.

(iii) For each such cell, Taylor coefficients through an order N_J for the local series of N_ρ, W_ρ, and Φ_ρ, together with the coefficients $P_{J,n}$ and $Q_{J,n}$ generated by Theorem III.11 and hence the residual coefficients $C^{(k)}_{\rho,J,n}$.

(iv) A branch certificate: the branch of $\log N_\rho$ on the disk is the analytic continuation of the real logarithm at the center c_J, and the nonvanishing of N_ρ is certified, for example, by a Rouché-type lower bound

$$|a_{J,0}| - \sum_{n=1}^{N_J} |a_{J,n}| r_J^n - E_J^N(r_J) > 0, \tag{III.52}$$

where $E_J^N(r_J)$ is a certified tail bound for the Taylor series of N_ρ on $|u| \le r_J$.

(v) Majorants $M_{J,k}$ such that $|G_{k,\rho}(c_J + u)| \le M_{J,k}$ for $|u| \le r_J$, and tail constants $C_k, c_k > 0$ such that

$$|G_{k,\rho}(t)| \le C_k(1 + |t|)e^{-c_k|t|}, \qquad |t| \ge L. \tag{III.53}$$

The certificate is a posteriori: it can be verified from interval or ball arithmetic applied to the displayed Taylor series, to the hypergeometric residue classes, and to the explicit endpoint estimates of Lemma III.12.

Given such a certificate, define the finite approximation

$$\widehat{\mathcal{R}}_{k,\rho} = \sum_{J \in \mathfrak{A}_{\rho,L}} \sum_{n=0}^{N_J} C^{(k)}_{\rho,J,n} \Omega_{J,n}. \tag{III.54}$$

The cell remainder is controlled only by the ratio between the integration half-length and the analytic radius. If $q_J = h_J/r_J < 1$, Cauchy's estimate gives

$$E_{J,k}(N_J) \le 2h_J M_{J,k} \frac{q_J^{N_J+1}}{1 - q_J}. \tag{III.55}$$

The two tails are controlled by (III.53):

$$E_{\mathrm{tail},k}(L) \le 2C_k e^{-c_k L} \left(\frac{1 + L}{c_k} + \frac{1}{c_k^2} \right). \tag{III.56}$$

Thus the certificate proves the explicit error bound

$$|\mathcal{R}_{k,\rho} - \widehat{\mathcal{R}}_{k,\rho}| \le E_{\mathrm{tail},k}(L) + \sum_{J \in \mathfrak{A}_{\rho,L}} E_{J,k}(N_J). \tag{III.57}$$

Proposition III.19 (Effective coefficient construction). *For rational $\alpha = p/q \neq 1$, all ingredients of Definition III.18 are obtained from explicit recurrences and finite algebraic operations once the dyadic atlas and truncation orders are chosen.*

Proof. The half-line kernel H_ρ is a finite sum of hypergeometric residue classes by Theorems II.7 and II.8. Each summand has its standard hypergeometric differential equation; a common annihilating equation for H_ρ is obtained by taking a least common left multiple, or equivalently by working componentwise and summing the resulting Taylor series. The logit substitution $z(t) = \kappa_\rho^{-1/p} e^{t/p}$ and the gauge division by $B_\rho(t) = H_{0,\rho}(1+e^t)^{-\nu}$ give the local Taylor series of N_ρ by ordinary composition and division. The explicit functions W_ρ and Φ_ρ in (III.37) have elementary Taylor recurrences.

Once the coefficients of N_ρ are known on a cell, Theorem III.11 gives the coefficients of $N_\rho \log N_\rho$ and $N_\rho(\log N_\rho)^2$ with the branch fixed by the real positive value $N_\rho(c_J) > 0$. The convolution formulas (III.47)–(III.48) then give the coefficients of the residual integrands. The zero-free test (III.52), the Cauchy remainder (III.55), and the tail bound (III.56) are finite inequalities. Hence the computation is reducible to finite coefficient recurrences and finite majorant checks. ☐

Remark III.20 (Finding versus checking a certificate). Lemma III.14 proves that zero-free dyadic atlases with summable majorants exist: the tails are zero-free because $N_\rho(t) \to 1$ in a complex strip, and only finitely many central cells may require refinement. Definition III.18 separates the search for such cells from the verification of a computed value. A numerical implementation may refine cells until the lower bound (III.52) and the requested error budget in (III.57) are both satisfied; the reported value is then accompanied by checkable branch, radius, truncation, and tail data.

III-G Final assembly

Define the half-line sums

$$\mathcal{B}_1 = \mathcal{B}_{1,\rho_+} + \mathcal{B}_{1,\rho_-}, \qquad \mathcal{B}_2 = \mathcal{B}_{2,\rho_+} + \mathcal{B}_{2,\rho_-}, \tag{III.58}$$

$$\mathcal{R}_1 = \mathcal{R}_{1,\rho_+} + \mathcal{R}_{1,\rho_-}, \qquad \mathcal{R}_2 = \mathcal{R}_{2,\rho_+} + \mathcal{R}_{2,\rho_-}. \tag{III.59}$$

Then

$$R'(1) = \mathcal{B}_1 + \mathcal{R}_1, \qquad R''(1) = \mathcal{B}_2 + \mathcal{R}_2.$$

Theorem III.21 (Constructive rational-index stable varentropy)**.** *Let* $\alpha = p/q \in (0,2) \setminus \{1\}$ *be rational and let* β *be a conventional stable skewness parameter whose corresponding phase* b*, obtained from* (II.4)*, satisfies the strict non-extremal admissibility condition* $|b| < b_{\max}(\alpha)$*. Define* $\rho_\pm$ *by* (II.8)*. Then*

$$\mathrm{VE}_{\alpha,\beta} = \frac{\mathcal{B}_2 + \mathcal{R}_2}{\alpha} - \left(\frac{\mathcal{B}_1 + \mathcal{R}_1}{\alpha}\right)^2. \tag{III.60}$$

Equivalently,

$$\mathrm{VE}_{\text{factor}} = \frac{\mathcal{B}_2}{\alpha} - \left(\frac{\mathcal{B}_1}{\alpha}\right)^2, \tag{III.61}$$

$$\mathrm{VE}_{\text{residual}} = \frac{\mathcal{R}_2}{\alpha} - \frac{2\mathcal{B}_1\mathcal{R}_1 + \mathcal{R}_1^2}{\alpha^2}, \tag{III.62}$$

and

$$\mathrm{VE}_{\alpha,\beta} = \mathrm{VE}_{\text{factor}} + \mathrm{VE}_{\text{residual}}. \tag{III.63}$$

Proof. Equations (III.58) and (III.59) give

$$R'(1) = \sum_\rho \int H_\rho \log H_\rho = \mathcal{B}_1 + \mathcal{R}_1,$$

$$R''(1) = \sum_\rho \int H_\rho (\log H_\rho)^2 = \mathcal{B}_2 + \mathcal{R}_2.$$

Substituting these into the kernel-level formula (III.2) gives (III.60). Expanding (III.60) into factor contribution and residual contribution gives (III.61)–(III.63). ☐

III-H Edge cases and the skewed $\alpha = 1$ obstruction

The formula above is stated for rational $\alpha \neq 1$ in the strict non-extremal range. The two classical boundary checks are still best understood through the same factorization idea: when the half-line kernel itself is hyperexponential, choose the endpoint-matched factor to be the kernel, so that the residual is identically one.

Proposition III.22 (Gaussian endpoint through the factorization)**.** *At $\alpha = 2$ and $b = 0$,*

$$\mathrm{VE}_{2,0} = \frac{1}{2}.$$

Proof. The half-line kernel is

$$H(z) = K_{2,1/2}(z) = \pi^{-1/2} e^{-z^2/4}.$$

It is hyperexponential, since $H'/H = -z/2$. Therefore the factorization can be chosen as

$$B = H, \qquad N = 1.$$

The residual terms vanish identically. The reflected kernel power integral is

$$R(\lambda) = 2\int_0^\infty H(z)^\lambda \, \mathrm{d}z = 2\pi^{-\lambda/2}\int_0^\infty e^{-\lambda z^2/4} \, \mathrm{d}z = 2\pi^{(1-\lambda)/2}\lambda^{-1/2}.$$

Hence

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \log R(\lambda)\bigg|_{\lambda=1} = \frac{1}{2}.$$

By the scale-cancellation theorem this is the varentropy. ☐

Proposition III.23 (Cauchy transition family through the factorization)**.** *At $\alpha = 1$, the transition half-line kernel*

$$K_{1,\rho}(z) = \frac{\sin(\pi\rho)}{\pi(1 + 2z\cos(\pi\rho) + z^2)}, \qquad 0 < \rho < 1,$$

is hyperexponential. The reflected kernel curvature is independent of ρ, and

$$\mathrm{VE}_1 = \frac{\pi^2}{3}.$$

In particular the symmetric Cauchy point $\rho = 1/2$, equivalently $b = 0$, has $\mathrm{VE}_{1,0} = \pi^2/3$.

Proof. Put $\theta = \pi\rho$. Then

$$H(z) = K_{1,\rho}(z) = \frac{\sin\theta}{\pi(1 + 2z\cos\theta + z^2)}$$

and

$$\frac{H'(z)}{H(z)} = -\frac{2z + 2\cos\theta}{z^2 + 2z\cos\theta + 1} \in \mathbb{C}(z).$$

Thus choose $B = H$ and $N = 1$; the residual disappears. The side power integral is

$$I_\rho(\lambda) = \int_0^\infty K_{1,\rho}(z)^\lambda \, \mathrm{d}z = \pi^{-\lambda}\sin(\theta)^{1-\lambda}\int_0^\theta \sin^{2\lambda-2} u \, \mathrm{d}u.$$

Adding the reflected side $\rho \mapsto 1 - \rho$ gives

$$R(\lambda) = \pi^{-\lambda}\sin(\theta)^{1-\lambda}\int_0^\pi \sin^{2\lambda-2} u \, \mathrm{d}u = \pi^{-\lambda}\sin(\theta)^{1-\lambda}\sqrt{\pi}\,\frac{\Gamma(\lambda - 1/2)}{\Gamma(\lambda)}.$$

The θ-dependent factor contributes only an affine term to $\log R(\lambda)$, hence no second derivative. Therefore

$$\mathrm{VE}_1 = (\log R)''(1) = \psi_1(1/2) - \psi_1(1) = \frac{\pi^2}{2} - \frac{\pi^2}{6} = \frac{\pi^2}{3}.$$

☐

Remark III.24. The skewed $\alpha = 1$ law is a different edge case. The preceding proposition concerns the $\alpha = 1$ transition kernel that arises as the limit of the phase/Fox–H half-line representation. After the two reflected half-lines are assembled, it is a location-scale Cauchy law. Indeed, with $\theta = \pi\rho$ and the corresponding scale factor $A = \csc\theta$, the density becomes

$$f_\rho(x) = \frac{1}{\pi\{1 + (x + \cot\theta)^2\}},$$

up to the harmless location convention. Its varentropy is therefore $\pi^2/3$, as Proposition III.23 shows through the kernel-power calculation.

This is not the same object as the skewed S_1-parameterized $\alpha = 1$ stable law with Nolan skewness $\beta \neq 0$. In the usual standardized S_1 convention,

$$\varphi_\beta(t) = \exp\left\{-|t|\left(1 + \mathrm{i}\frac{2\beta}{\pi}\,\mathrm{sgn}(t)\log|t|\right)\right\}, \qquad -1 < \beta < 1, \tag{III.64}$$

up to location and scale conventions; see Nolan [7]. Fourier inversion gives

$$f_{1,\beta}(x) = \frac{1}{\pi}\Re\int_0^\infty e^{-u}e^{-\mathrm{i}\gamma u\log u}e^{-\mathrm{i}xu}\,\mathrm{d}u, \qquad \gamma = \frac{2\beta}{\pi}. \tag{III.65}$$

Equivalently,

$$f_{1,\beta}(x) = \frac{1}{\pi}\int_0^\infty e^{-u}\cos\left(xu + \frac{2\beta}{\pi}u\log u\right)\,\mathrm{d}u.$$

The logarithmic term $u\log u$ is precisely what is absent from the Cauchy transition kernel. The following results show that this is not merely an inconvenience: it is a structural obstruction to the holonomic/gamma-product method used for rational $\alpha \neq 1$.

Lemma III.25 (The logarithmic Fourier kernel is not holonomic)**.** *Let*

$$g_\gamma(u) = e^{-u}e^{-\mathrm{i}\gamma u\log u}, \qquad \gamma \neq 0.$$

Then g_γ is not holonomic over $\mathbb{C}(u)$. Equivalently, it satisfies no nonzero linear differential equation with rational-function coefficients.

Proof. Analytically continue a local branch of g_γ once around $u = 0$. Since $\log u$ changes to $\log u + 2\pi\mathrm{i}$, the continued branch is

$$e^{-u}e^{-\mathrm{i}\gamma u(\log u + 2\pi\mathrm{i})} = e^{2\pi\gamma u}g_\gamma(u).$$

After m turns, the branch is

$$e^{2\pi\gamma mu}g_\gamma(u), \qquad m \in \mathbb{Z}.$$

If g_γ were holonomic of order N, all analytic continuations of a local germ would lie in the same N-dimensional solution space. Hence the family

$$\{e^{2\pi\gamma mu}g_\gamma(u) \,:\, m \in \mathbb{Z}\}$$

would have finite-dimensional span. Since g_γ is not identically zero, this would imply that

$$\{e^{2\pi\gamma mu} \,:\, m \in \mathbb{Z}\}$$

has finite-dimensional span. Distinct exponentials have linearly independent germs, and $2\pi\gamma m$ are distinct as m varies because $\gamma \neq 0$. This contradiction proves the claim. ☐

Proposition III.26 (The Fourier-kernel Mellin transform is not a finite gamma product)**.** *Define*

$$\mathcal{G}_\gamma(a) = \int_0^\infty u^{a-1}e^{-u}e^{-\mathrm{i}\gamma u\log u}\,\mathrm{d}u, \qquad \Re a > 0.$$

If $\gamma \neq 0$, then the meromorphic continuation of $\mathcal{G}_\gamma$ has poles of unbounded order at the nonpositive integers. Consequently $\mathcal{G}_\gamma$ is neither a finite gamma product nor a finite linear combination of gamma products with affine arguments.

Proof. Near $u = 0$,

$$e^{-u}e^{-\mathrm{i}\gamma u\log u} = \exp(-u - \mathrm{i}\gamma u\log u) = \sum_{n=0}^{\infty} \frac{u^n}{n!}(-1 - \mathrm{i}\gamma\log u)^n.$$

For fixed n, the coefficient of $u^n(\log u)^n$ is $(-\mathrm{i}\gamma)^n/n!$, which is nonzero for $\gamma \neq 0$. Also

$$\int_0^1 u^{a+n-1}(\log u)^k \,\mathrm{d}u = (-1)^k \frac{k!}{(a+n)^{k+1}}.$$

Thus the term $(-\mathrm{i}\gamma)^n u^n(\log u)^n/n!$ contributes a pole of order $n+1$ at $a = -n$. After subtracting finitely many terms of the displayed local expansion, the remainder is $O(u^{N+1}(1+|\log u|)^{N+1})$, so the standard Mellin-continuation argument shows that these are the exact pole orders. Hence the pole order at $a = -n$ is $n+1$, and the pole orders are unbounded.

A finite gamma product of the form

$$Ce^{\lambda a}\prod_{j=1}^{J}\Gamma(A_j a + B_j)^{m_j}$$

has pole orders bounded by the finite number $\sum_j |m_j|$, after ignoring factors that contribute zeros rather than poles. A finite linear combination of such products also has pole orders bounded by a finite constant, because cancellation can only reduce pole order or combine finitely many bounded pole orders. Therefore $\mathcal{G}_\gamma$ cannot be a finite gamma product or a finite linear combination of gamma products. □

Theorem III.27 (The conventional skewed $\alpha = 1$ density is not holonomic)**.** *Let $f_{1,\beta}$ be the conventional standardized S_1-parameterized stable density with $-1 < \beta < 1$ and $\beta \neq 0$. Then $f_{1,\beta}$ is not holonomic on either tail.*

Proof. Put $\gamma = 2\beta/\pi \neq 0$, and use the Fourier representation (III.65). Let

$$I(x) = \int_0^\infty e^{-u}e^{-\mathrm{i}\gamma u\log u}e^{-\mathrm{i}xu}\,\mathrm{d}u.$$

The coefficient of $u^n(\log u)^k$ in the small-u expansion of the Fourier kernel is

$$a_{n,k} = \frac{1}{n!}\binom{n}{k}(-1)^{n-k}(-\mathrm{i}\gamma)^k.$$

Moreover, for $a > -1$,

$$\int_0^\infty e^{-\mathrm{i}xu}u^a\,\mathrm{d}u = \Gamma(a+1)e^{-\mathrm{i}\pi(a+1)/2}x^{-a-1}, \qquad x > 0,$$

interpreted by the usual oscillatory regularization. Hence

$$\int_0^\infty e^{-\mathrm{i}xu}u^n(\log u)^k\,\mathrm{d}u = \frac{\partial^k}{\partial a^k}\left[\Gamma(a+1)e^{-\mathrm{i}\pi(a+1)/2}x^{-a-1}\right]\Big|_{a=n}.$$

Watson's lemma for Fourier integrals then gives a full tail expansion

$$f_{1,\beta}(x) \sim \sum_{n\geq 1} x^{-n-1}P_n(\log x), \qquad x \to +\infty,$$

where P_n is a polynomial. The top $(\log x)^n$ contribution from the $k = n$ term is purely imaginary and disappears after taking the real part in (III.65). The next coefficient, that of $(\log x)^{n-1}$, receives contributions from $k = n$ and $k = n-1$. A direct differentiation of the previous display gives

$$\Re\,[x^{-n-1}(\log x)^{n-1}]\,I(x) = n\gamma^{n-1}\left(1 + \frac{\pi\gamma}{2}\right).$$

Since $\pi\gamma/2 = \beta$,

$$[x^{-n-1}(\log x)^{n-1}]\,f_{1,\beta}(x) = \frac{n}{\pi}\gamma^{n-1}(1+\beta). \tag{III.66}$$

For $-1 < \beta < 1$ and $\beta \neq 0$, the coefficient in (III.66) is nonzero for every $n \geq 1$. Thus the positive tail expansion contains logarithmic powers of unbounded degree.

A holonomic function, as a solution of a finite-order linear differential equation with rational coefficients, has at infinity a finite formal asymptotic structure: a finite sum of exponential factors times power series, with logarithmic powers bounded by the order of the equation. This is the standard Levelt–Turrittin asymptotic form for linear differential equations with rational coefficients. Therefore a holonomic function cannot have an algebraic tail expansion whose logarithmic powers have unbounded degree. Hence $f_{1,\beta}$ is not holonomic on the positive tail.

The negative tail is the same calculation with $x \to -\infty$. The coefficient in (III.66) is then replaced by $n\gamma^{n-1}(1-\beta)/\pi$, again nonzero in the non-extremal range. Thus the density is not holonomic on either tail. □

Corollary III.28 (Half-line Mellin transforms are not finite gamma products)**.** *Let*

$$M_+(s) = \int_0^\infty x^{s-1} f_{1,\beta}(x)\ \mathrm{d}x, \qquad M_-(s) = \int_0^\infty x^{s-1} f_{1,\beta}(-x)\ \mathrm{d}x,$$

initially in the common Mellin strip. If $-1 < \beta < 1$ and $\beta \neq 0$, then the meromorphic continuations of M_+ and M_- have poles of unbounded order. Consequently neither half-line Mellin transform is a finite gamma product or a finite linear combination of gamma products with affine arguments.

Proof. A term

$$c_n x^{-n-1}(\log x)^{n-1}, \qquad c_n \neq 0,$$

in the positive-tail expansion contributes to $M_+(s)$ the model integral

$$c_n \int_1^\infty x^{s-n-2}(\log x)^{n-1}\ \mathrm{d}x = c_n \frac{(n-1)!}{(n+1-s)^n},$$

up to the harmless sign convention determined by analytic continuation. Thus M_+ has a pole of order n at $s = n+1$ for every $n \geq 1$. The same argument applies to M_-, using the nonzero coefficients with $1-\beta$. Finite gamma products and finite sums of such products have uniformly bounded pole orders, as in Proposition III.26. Hence neither M_+ nor M_- can be represented by such a finite gamma-product expression. ☐

III-I Numerical use and interpolation for floating-point parameters

The formula of Theorem III.21 is constructive on rational strata $\alpha = p/q$. In software, however, the user usually supplies floating-point parameters (α, β). The natural role of the rational-index formula is therefore not to approximate every input α by a large denominator rational at runtime. Rather, the rational formula should be used to generate high-precision values at selected rational nodes; those values then define an interpolation table for real parameters.

There is a useful normalization of the skewness coordinate. For $\alpha \neq 1$, first convert the conventional skewness β to the phase coordinate b by (II.4). Then set

$$u = \frac{b}{b_{\max}(\alpha)}, \qquad \eta = u^2. \tag{III.67}$$

In the strict non-extremal range, $|u| < 1$, so $0 \leq \eta < 1$. The square is useful because reflection of the density gives

$$\mathrm{VE}_{\alpha,b} = \mathrm{VE}_{\alpha,-b},$$

so the varentropy is an even function of the phase coordinate.

The quantities to interpolate should be the normalized logarithmic moments, not the final quadratic combination. Define

$$m_1(\alpha,\eta) = \frac{\mathcal{B}_1 + \mathcal{R}_1}{\alpha}, \qquad m_2(\alpha,\eta) = \frac{\mathcal{B}_2 + \mathcal{R}_2}{\alpha}. \tag{III.68}$$

Then

$$\mathrm{VE}_{\alpha,\beta} = m_2(\alpha,\eta) - m_1(\alpha,\eta)^2. \tag{III.69}$$

Interpolating m_1 and m_2 separately preserves the identity $R(1) = \alpha$ in the normalization and avoids mixing interpolation error with the nonlinear squaring operation.

The parameter domain should be split into blocks that do not cross the singular point $\alpha = 1$:

$$0 < \alpha < 1, \qquad 1 < \alpha < 2. \tag{III.70}$$

On a compact block

$$\alpha \in [\alpha_L, \alpha_U] \subset (0,1) \quad \text{or} \quad \alpha \in [\alpha_L, \alpha_U] \subset (1,2), \qquad 0 \leq \eta \leq \eta_{\max} < 1,$$

the half-line kernels, endpoint-matched factors, and canonical residual-atlas integrands vary smoothly in the real parameters. Thus Chebyshev or barycentric interpolation in (α, η) is the appropriate numerical layer.

A practical table-building procedure is as follows.

1. Choose interpolation nodes in the α-block, preferably Chebyshev nodes. Replace each node by a nearby rational value $\tilde{\alpha}_i = p_i/q_i$ with a prescribed denominator bound $q_i \leq Q_{\max}$, and use the actual rational nodes in the interpolation. This prevents expression sizes from exploding.

2. Choose Chebyshev nodes η_j in $[0, \eta_{\max}]$. The corresponding phase values are $b_{ij} = b_{\max}(\tilde{\alpha}_i)\sqrt{\eta_j}$; the sign is irrelevant for varentropy.
3. At each node $(\tilde{\alpha}_i, \eta_j)$, compute
$$m_1(\tilde{\alpha}_i, \eta_j), \qquad m_2(\tilde{\alpha}_i, \eta_j)$$
from the exact factor moments and the zero-free dyadic-atlas residual formula (III.50)–(III.51), using the certificate structure of Subsection III-F.
4. Fit tensor-product Chebyshev coefficients, or store a barycentric interpolation table, for both m_1 and m_2.
5. Validate the table on rational holdout points not used in the fit.

The residual part should be evaluated using the zero-free dyadic logit atlas of Subsection III-E. Outside a finite central block the cells are the fixed unit cells
$$I_\ell = \left[\ell - \frac{1}{2}, \ell + \frac{1}{2}\right], \qquad \ell \in \mathbb{Z},$$
while only finitely many central cells may be subdivided. The tail sum over ℓ is truncated only by a requested tolerance. This is intrinsic: by Lemma III.14, the cell tails are exponentially damped as $\ell \to \pm\infty$, up to the harmless linear factor in the second residual moment.

At runtime, one can implement a simple procedure. Given (α, β), first handle the exact anchors
$$\mathrm{VE}_{2,0} = \frac{1}{2}, \qquad \mathrm{VE}_{1,0} = \frac{\pi^2}{3}.$$
For $\alpha \neq 1$, convert β to b using the branch convention in (II.2)–(II.4), then to η by (III.67), choose the appropriate interpolation block, evaluate the two interpolants m_1, m_2, and return (III.69). The conventional skewed $\alpha = 1$, $\beta \neq 0$, case is not handled by this symbolic interpolant for the structural reasons explained in Subsection III-H; it would require a separate numerical method.

The basic validation checks are built into the structure:
$$\mathrm{VE}_{\alpha,\beta} = \mathrm{VE}_{\alpha,-\beta}, \qquad R(1) = \alpha, \qquad \mathrm{VE}_{2,0} = \frac{1}{2}, \qquad \mathrm{VE}_{1,0} = \frac{\pi^2}{3}. \tag{III.71}$$
In addition, one should compare nested interpolation degrees and evaluate the rational formula at held-out rational nodes. This turns the exact rational-index formula into a practical evaluator for real floating-point parameters while keeping the symbolic construction and the numerical interpolation layer conceptually separate.

IV Varentropy convergence in the non-extremal stable CLT

This section records a convergence theorem for varentropy in the style of an entropic central limit theorem. It is independent of rationality of α.

IV-A Setup

If a random variable X has density f on $\mathbb{R}$, define
$$h(X) := -\int_{\mathbb{R}} f(x) \log f(x)\, \mathrm{d}x, \qquad \mathrm{VE}(X) := \mathrm{Var}[-\log f(X)].$$
Let
$$M_2(f) := \int_{\mathbb{R}} f(x)(\log f(x))^2\, \mathrm{d}x.$$
Whenever $M_2(f) < \infty$,
$$\mathrm{VE}(X) = M_2(f) - h(X)^2. \tag{IV.1}$$

Throughout this section, Z denotes a non-extremal stable law: either Gaussian, or stable with $0 < \alpha < 2$ and $-1 < \beta < 1$. Its density ψ is positive and continuous on $\mathbb{R}$, and in the non-Gaussian case satisfies
$$\psi(x) \sim c_-|x|^{-(1+\alpha)} \quad (x \to -\infty), \qquad \psi(x) \sim c_+ x^{-(1+\alpha)} \quad (x \to +\infty). \tag{IV.2}$$
We use this standard non-extremal stable tail form as in Bobkov–Chistyakov–Götze [8, Definition 1.2 and Eq. (1.2)].

IV-B An abstract continuity theorem

Lemma IV.1 (Finite varentropy of the target). *Let Z have a non-extremal stable density ψ. Then*

$$\int_{\mathbb{R}} \psi(x)|\log \psi(x)|\ \mathrm{d}x < \infty, \qquad \int_{\mathbb{R}} \psi(x)(\log \psi(x))^2\ \mathrm{d}x < \infty.$$

Proof. The tail estimate (IV.2) gives

$$\psi(x)|\log \psi(x)|^k \asymp |x|^{-(1+\alpha)}(\log |x|)^k, \qquad |x| \to \infty,$$

for $k = 1, 2$. This is integrable for $\alpha > 0$. On compact sets ψ is positive and continuous, hence bounded above and below away from zero. The Gaussian case is immediate. ☐

Lemma IV.2 (Compact convergence of self-information moments). *Let ψ be a positive continuous density and let p_n be densities with $p_n \to \psi$ uniformly on compact sets. For every $T > 0$ and $k = 1, 2$,*

$$\int_{|x|\le T} p_n(x)(\log p_n(x))^k\ \mathrm{d}x \to \int_{|x|\le T} \psi(x)(\log \psi(x))^k\ \mathrm{d}x.$$

The same holds with $|\log p_n(x)|$ in place of $(\log p_n(x))^k$.

Proof. On $[-T, T]$, positivity and continuity give $0 < m_T \le \psi \le M_T < \infty$. Uniform convergence implies $m_T/2 \le p_n \le M_T + 1$ for large n. The functions $u \mapsto u(\log u)^k$ and $u \mapsto u|\log u|$ are Lipschitz on this compact interval, so the corresponding integrands converge uniformly on $[-T, T]$. ☐

Lemma IV.3 (Uniform tail control). *Suppose densities p_n satisfy*

$$\sup_n \|p_n\|_\infty \le M < \infty$$

and for some $\delta > 0$,

$$\sup_n \mathbb{E}|Z_n|^\delta \le C_\delta, \qquad Z_n \sim p_n.$$

Then

$$\sup_n \int_{|x|>T} p_n(x)(\log p_n(x))^2\ \mathrm{d}x \to 0 \qquad (T \to \infty).$$

Consequently the same is true with $|\log p_n(x)|$.

Proof. Fix $T \ge 2$ and split

$$A_n(T) = \{|x| > T : p_n(x) \le |x|^{-4}\}, \qquad B_n(T) = \{|x| > T : p_n(x) > |x|^{-4}\}.$$

For $0 < u \le 1$, $u(\log u)^2 \le 4u^{1/2}$. Hence on $A_n(T)$,

$$p_n(\log p_n)^2 \le 4|x|^{-2},$$

and the integral over $A_n(T)$ is at most $8/T$. On $B_n(T)$,

$$|\log p_n(x)| \le |\log M| + 4 \log |x|.$$

Thus

$$\int_{B_n(T)} p_n(\log p_n)^2 \le K_M \int_{|x|>T} p_n(x)(1 + \log^2 |x|)\ \mathrm{d}x.$$

By Markov's inequality, $\int_{|x|>T} p_n \le C_\delta T^{-\delta}$. Also $\log^2 r \le L_\delta r^{\delta/2}$ for $r \ge 1$, so for $|x| > T$,

$$\log^2 |x| \le L_\delta T^{-\delta/2}|x|^\delta.$$

The moment bound finishes the estimate. The first-moment logarithmic tail follows from $|y| \le (1 + y^2)/2$. ☐

Theorem IV.4 (Varentropy continuity). *Let Z_n have densities p_n and let Z have density ψ. Assume:*

(A1) *ψ is positive and continuous;*
(A2) *$p_n \to \psi$ uniformly on compact subsets of $\mathbb{R}$;*
(A3) *$\sup_n \|p_n\|_\infty < \infty$;*

(A4) $\sup_n \mathbb{E}|Z_n|^\delta < \infty$ *for some* $\delta > 0$*;*
(A5) $\int \psi(\log \psi)^2 < \infty$.

Then

$$h(Z_n) \to h(Z), \qquad M_2(p_n) \to M_2(\psi), \qquad \mathrm{VE}(Z_n) \to \mathrm{VE}(Z).$$

Proof. Choose T so that the target tail of $\psi(\log\psi)^2$ is small. Lemma IV.3 makes the tails of $p_n(\log p_n)^2$ uniformly small, and also controls the first-log tails. Lemma IV.2 gives convergence of both first and second self-information integrals on $[-T, T]$. Letting the tail error go to zero proves convergence of h and M_2. Then (IV.1) gives varentropy convergence. ☐

Remark IV.5. The tail estimate in Lemma IV.3 is a cleaned-up version of the tail decomposition used by Bobkov–Chistyakov–Götze in the proof of Lemma 6.1 of [8].

IV-C Bobkov's input package and the stable-law corollary

Let

$$Z_n = \frac{X_1 + \cdots + X_n}{b_n} - a_n$$

with X_i i.i.d., and suppose $Z_n \Rightarrow Z$, where Z is non-extremal stable. Denote densities by p_n and ψ whenever they exist.

Proposition IV.6 (Bobkov continuity package)**.** *Assume* $Z_n \Rightarrow Z$ *for a non-extremal stable law and* $I(Z_{n_0}) < \infty$ *for some* n_0*. Then:*

(B1) $\sup_{n \ge n_0} I(Z_n) < \infty$*;*
(B2) p_n *exists for all* $n \ge n_0$*, and for all sufficiently large* n *the density* p_n *is bounded and continuous;*
(B3) $\sup_x |p_n(x) - \psi(x)| \to 0$*;*
(B4) $\sup_{n \ge n_0} \|p_n\|_\infty < \infty$*;*
(B5) *for every* $0 < \delta < \alpha$, $\sup_n \mathbb{E}|Z_n|^\delta < \infty$.

Proof. Statement (B1) is Lemma 8.1 of [8]. The Fisher bound implies decay of the characteristic function through [8, (3.3)]. This supplies the hypotheses of Propositions 5.1 and 5.2 of [8], giving bounded continuous densities, differentiability for large n, and the uniform local limit theorem, which is (B2)–(B3). The sup-norm bound (B4) follows from the Fisher-information inequality in [8, (3.3)]. Finally, (B5) is the standard fractional-moment bound for normalized sums in the domain of attraction of a stable law, cited in [8, p. 1636] from [9]. ☐

Theorem IV.7 (Varentropy convergence in the non-extremal stable CLT)**.** *Assume*

$$Z_n = \frac{X_1 + \cdots + X_n}{b_n} - a_n \Rightarrow Z,$$

where Z *has a non-extremal stable law. If* $I(Z_{n_0}) < \infty$ *for some* n_0*, then*

$$h(Z_n) \to h(Z), \qquad M_2(p_n) \to M_2(\psi), \qquad \mathrm{VE}(Z_n) \to \mathrm{VE}(Z).$$

Proof. By Proposition IV.6, assumptions (A2)–(A4) of Theorem IV.4 hold after discarding finitely many initial indices; this does not affect convergence. Assumption (A1) holds because non-extremal stable densities are positive and continuous on $\mathbb{R}$. Assumption (A5) follows from Lemma IV.1. Theorem IV.4 gives the result. ☐

Corollary IV.8 (Bounded-variation sufficient condition)**.** *Assume the hypotheses of Theorem IV.7. If* X_1 *has an absolutely continuous density of bounded variation, then*

$$\mathrm{VE}(Z_n) \to \mathrm{VE}(Z).$$

Proof. If the density of X_1 has bounded variation, then $I_1(X_1) < \infty$ in the notation of Bobkov's *Moments of the Scores* [10]. Applying [10, Theorem I.3 with $k = 2$] to $S_3 = X_1 + X_2 + X_3$ gives $I(S_3) < \infty$. Since $Z_3 = S_3/b_3 - a_3$, scaling gives $I(Z_3) = b_3^2 I(S_3) < \infty$. Theorem IV.7 applies with $n_0 = 3$. ☐

Remark IV.9. Theorem IV.7 is a second-order analogue of the entropy convergence theorem of Bobkov–Chistyakov–Götze [11]. It does not give a convergence rate; rates would require quantitative local-limit estimates and quantitative second self-information tail bounds.

V Appendix

A Mellin kernels and factor residue formulas

Lemma A.1 (Mellin kernels for factor logarithms)**.** *For* $-1 < \Re s < 0$*,*

$$\int_0^\infty x^{s-1}\log(1+x)\,\mathrm{d}x = \frac{\pi}{s\sin(\pi s)}, \tag{A.1}$$

$$\int_0^\infty x^{s-1}\log^2(1+x)\,\mathrm{d}x = -\frac{2\pi(\psi(-s)+\gamma_E)}{s\sin(\pi s)}. \tag{A.2}$$

Proof. For $0 < \Re a < \Re t$, Euler's beta integral gives

$$\int_0^\infty x^{a-1}(1+x)^{-t}\,\mathrm{d}x = \frac{\Gamma(a)\Gamma(t-a)}{\Gamma(t)},$$

see the DLMF beta-function integral [12, Eq. 5.12.3]. Differentiate once and twice with respect to t at $t = 0$, after analytic continuation in t, and put $a = s$. Euler's reflection formula and $\Gamma(1-s) = -s\Gamma(-s)$ give (A.1) and (A.2). ☐

B Tail left-residue factor derivatives

This appendix records the tail analogue used when $0 < \alpha = p/q < 1$, $p < q$. The final form is parallel to the central formula in Theorem III.10: ordinary hypergeometric functions are used for the nonlogarithmic classes, and explicit Pochhammer–polygamma sums are used for the logarithmic classes.

Set

$$\theta = \pi p\rho, \qquad \Omega_\varepsilon = (-1)^{q+1}\frac{p^p}{q^q}\,\kappa\,e^{-\varepsilon \mathrm{i}\theta}, \qquad \varepsilon = \pm 1.$$

For $r = 1, \dots, q-1$, let

$$\delta_r = \frac{r}{q},$$

and define the parameter lists

$$\mathbf{A}_r = \left(\delta_r, \delta_r + \frac{1}{p}, \delta_r + \frac{2}{p}, \dots, \delta_r + 1\right), \tag{B.1}$$

$$\mathbf{B}_r = \left(\frac{r+1}{q}, \frac{r+2}{q}, \dots, \frac{r+q}{q}\right) \setminus \{1\} \;\cup\; \{1+\delta_r\}. \tag{B.2}$$

The repeated parameter $1+\delta_r$ may be cancelled in ordinary hypergeometric terms, but it is harmless to leave it present; the logarithmic class below differentiates the distinguished first numerator parameter δ_r. Define

$$C_r = \frac{(-1)^r\kappa^{\delta_r}\Gamma(1+p\delta_r)\Gamma(\delta_r)\Gamma(-\delta_r)}{\pi q\Gamma(r+1)}. \tag{B.3}$$

For $\varepsilon = \pm 1$ put

$$P_{r,\varepsilon} = \frac{\varepsilon C_r e^{-\varepsilon \mathrm{i}\theta\delta_r}}{2\mathrm{i}}. \tag{B.4}$$

Finally define the explicit Pochhammer–polygamma logarithmic class

$$\mathcal{D}_r(\Omega) = \sum_{n=1}^{\infty}\frac{\prod_{a\in\mathbf{A}_r}(a)_n}{\prod_{b\in\mathbf{B}_r}(b)_n\,n!}\Omega^n\left[\psi(\delta_r+n)-\psi(\delta_r)\right]. \tag{B.5}$$

Theorem B.1 (Tail left-residue factor derivatives)**.** *Assume* $0 < \alpha = p/q < 1$*, so* $p < q$*. The noninteger residue classes are*

$$F^{(1)}_{\mathrm{ni},\rho} = \sum_{r=1}^{q-1}\sum_{\varepsilon=\pm1} P_{r,\varepsilon}\,{}_{p+1}F_q\left(\begin{matrix}\mathbf{A}_r\\ \mathbf{B}_r\end{matrix};\Omega_\varepsilon\right), \tag{B.6}$$

$$F^{(2)}_{\mathrm{ni},\rho} = 2\sum_{r=1}^{q-1}\sum_{\varepsilon=\pm1} P_{r,\varepsilon}\left[(\gamma_E+\psi(\delta_r))\,{}_{p+1}F_q\left(\begin{matrix}\mathbf{A}_r\\ \mathbf{B}_r\end{matrix};\Omega_\varepsilon\right)+\mathcal{D}_r(\Omega_\varepsilon)\right]. \tag{B.7}$$

The integer collision class is represented by the convergent built-in sum below. For $n \geq 1$ *set*

$$S_n = \sin(-\pi p\rho n), \qquad C_n = \cos(\pi p\rho n), \qquad G_n = \Gamma(1+pn)\kappa^n,$$

$$h_N = H_N - \gamma_E,$$

where H_N and $H_N^{(2)}$ denote harmonic numbers. Define

$$I_n^{(1)} = \frac{(-1)^{(q+1)n} G_n}{\pi (qn)!}\left[\frac{S_n h_{qn}}{n} + \frac{1}{q}\left(\frac{[-p\psi(1+pn) - \log\kappa]S_n + \pi p\rho C_n}{n} + \frac{S_n}{n^2}\right)\right], \tag{B.8}$$

$$D_n = \frac{(-1)^{(q+1)n} G_n S_n}{\pi q n (qn)!}, \tag{B.9}$$

$$I_n^{(2)} = 2\left(H_{n-1} I_n^{(1)} - \psi_1(n) D_n\right). \tag{B.10}$$

Then

$$F_\rho'(0;\kappa) = F_{\mathrm{ni},\rho}^{(1)} + \sum_{n=1}^{\infty} I_n^{(1)}, \tag{B.11}$$

$$F_\rho''(0;\kappa) = F_{\mathrm{ni},\rho}^{(2)} + \sum_{n=1}^{\infty} I_n^{(2)}. \tag{B.12}$$

All series are ordinary convergent series in the tail regime $p < q$.

Proof. The left-residue contour has noninteger poles at $v = -n - r/q$, $r = 1, \dots, q-1$. The residue before summing over n is

$$\frac{(-1)^{qn+r}}{\pi q (qn+r)!}\Gamma(1 + p(n + r/q))\Gamma(-n - r/q)\Gamma(n + r/q)\sin(-\pi p\rho(n + r/q))\kappa^{n+r/q}.$$

Using

$$\Gamma(-\delta_r - n) = \frac{(-1)^n \Gamma(-\delta_r)}{(1+\delta_r)_n}, \qquad \Gamma(\delta_r + n) = \Gamma(\delta_r)(\delta_r)_n,$$

together with Gauss multiplication for the p- and q-step gamma factors gives precisely the hypergeometric coefficient with parameter lists (B.1)–(B.2). Splitting the sine into the two exponentials gives the phases $P_{r,\varepsilon}$ and the arguments Ω_ε, proving (B.6).

For the second derivative, the Mellin kernel has the additional factor $2(\gamma_E + \psi(n + \delta_r))$. Decompose

$$\gamma_E + \psi(n + \delta_r) = \gamma_E + \psi(\delta_r) + \left[\psi(n + \delta_r) - \psi(\delta_r)\right].$$

The first two terms multiply the ordinary hypergeometric class, and the bracket gives the Pochhammer–polygamma sum (B.5). This proves (B.7). The integer formulas (B.8)–(B.10) come from the double-pole collision of $\Gamma(qv)$ with $1/\sin(\pi v)$ at $v = -n$. Lemma C.2 gives the Laurent expansion explicitly and shows that, with the left-closure sign convention, the first-derivative contribution is $I_n^{(1)}$ and the second-derivative contribution is $I_n^{(2)}$. Because $p < q$, the noninteger hypergeometric classes have numerator count at most denominator count, and the integer class contains the superexponentially decaying ratio $\Gamma(1 + pn)/(qn)!$. Hence all series converge. ☐

C Laurent expansions for the integer collision classes

This subsection spells out the finite Laurent calculations used in Theorem III.10 and Theorem B.1. The only input is the standard gamma expansion, valid for every integer $m \geq 0$,

$$\Gamma(-m + \varepsilon) = \frac{(-1)^m}{m!}\left\{\frac{1}{\varepsilon} + \psi(m+1) + \frac{\varepsilon}{2}\left[\psi(m+1)^2 + 2\zeta(2) - \psi_1(m+1)\right] + O(\varepsilon^2)\right\}. \tag{C.1}$$

Here ψ and ψ_1 are the di- and trigamma functions. Since $\psi(m+1) = H_m - \gamma_E$ and $2\zeta(2) - \psi_1(m+1) = \zeta(2) + H_m^{(2)}$, this is the usual harmonic-number form of the expansion at a pole of Γ.

Lemma C.1 (Central integer-collision Laurent expansion)**.** *Assume $p > q$. Fix $N \geq 1$ and write*

$$v = N + w, \qquad L = \log\kappa, \qquad P = \pi p\rho.$$

Let $S_N = \sin(PN)$, $C_N = \cos(PN)$, and define

$$A_N = \frac{(-1)^{pN-1}}{\Gamma(pN)}, \qquad h_N^{(p)} = \psi(pN),$$

$$c_N^{(p)} = \frac{1}{2}\left(\psi(pN)^2 + 2\zeta(2) - \psi_1(pN)\right).$$

Then

$$\Gamma(1-pv) = a_{-1,N}w^{-1} + a_{0,N} + a_{1,N}w + O(w^2), \tag{C.2}$$

where

$$a_{-1,N} = -\frac{A_N}{p}, \qquad a_{0,N} = A_N h_N^{(p)}, \qquad a_{1,N} = -A_N p\, c_N^{(p)}.$$

Furthermore,

$$\frac{1}{\sin(\pi v)} = (-1)^N \left(\frac{1}{\pi w} + \frac{\pi w}{6} + O(w^3)\right), \tag{C.3}$$

while the analytic factor has the expansion

$$\frac{\Gamma(qv)\sin(Pv)\kappa^{-v}}{v} = g_{0,N} + g_{1,N}w + g_{2,N}w^2 + O(w^3), \tag{C.4}$$

with

$$G_N = \frac{\Gamma(qN)\kappa^{-N}}{N}, \qquad \ell_N = q\psi(qN) - L - \frac{1}{N}, \qquad b_N = q^2\psi_1(qN) + \frac{1}{N^2},$$
$$g_{0,N} = G_N S_N,$$
$$g_{1,N} = G_N(\ell_N S_N + PC_N),$$
$$g_{2,N} = \frac{G_N}{2}\left[(\ell_N^2 + b_N - P^2)S_N + 2\ell_N PC_N\right].$$

Consequently,

$$\frac{\Gamma(1-pv)\Gamma(qv)\sin(Pv)\kappa^{-v}}{v\sin(\pi v)} = d_{-2,N}w^{-2} + d_{-1,N}w^{-1} + d_{0,N} + O(w), \tag{C.5}$$

where

$$d_{-2,N} = \frac{(-1)^N}{\pi}a_{-1,N}g_{0,N},$$
$$d_{-1,N} = \frac{(-1)^N}{\pi}(a_{-1,N}g_{1,N} + a_{0,N}g_{0,N}),$$
$$d_{0,N} = \frac{(-1)^N}{\pi}(a_{-1,N}g_{2,N} + a_{0,N}g_{1,N} + a_{1,N}g_{0,N}) + (-1)^N\frac{\pi}{6}a_{-1,N}g_{0,N}.$$

Finally,

$$\psi(-N-w) + \gamma_E = \frac{1}{w} + H_N - (\zeta(2) + H_N^{(2)})w + O(w^2). \tag{C.6}$$

Thus the coefficient of w^{-1} in the product of (C.5) *and* (C.6) *is*

$$d_{0,N} + H_N d_{-1,N} - (\zeta(2) + H_N^{(2)})d_{-2,N}.$$

Proof. In (C.1), take $m = pN-1$ and $\varepsilon = -pw$. This gives (C.2). The sine expansion (C.3) follows from

$$\sin(\pi(N+w)) = (-1)^N\left(\pi w - \frac{\pi^3 w^3}{6} + O(w^5)\right).$$

For (C.4), write

$$\frac{\Gamma(q(N+w))\kappa^{-N-w}}{N+w} = G_N\left(1 + \ell_N w + \frac{\ell_N^2 + b_N}{2}w^2 + O(w^3)\right)$$

and

$$\sin(P(N+w)) = S_N + PC_N w - \frac{P^2 S_N}{2}w^2 + O(w^3).$$

Multiplication gives $g_{0,N}, g_{1,N}, g_{2,N}$. Multiplying (C.2), (C.3), and (C.4) gives (C.5). The expansion (C.6) follows from the pole expansion of ψ at $-N$; equivalently, differentiate the logarithm of (C.1) or use the recurrence for ψ. The final coefficient extraction is immediate. ☐

Lemma C.2 (Tail integer-collision Laurent expansion)**.** *Assume $p < q$. Fix $n \geq 1$ and write*

$$v = -n + w, \qquad L = \log \kappa, \qquad P = \pi p \rho.$$

Let

$$S_n = \sin(-Pn), \qquad C_n = \cos(Pn), \qquad G_n = \Gamma(1 + pn)\kappa^n,$$

and $h_{qn} = H_{qn} - \gamma_E = \psi(qn + 1)$. Then

$$\Gamma(qv) = \frac{(-1)^{qn}}{(qn)!}\left(\frac{1}{qw} + h_{qn} + O(w)\right), \tag{C.7}$$

$$\frac{1}{\sin(\pi v)} = (-1)^n \left(\frac{1}{\pi w} + \frac{\pi w}{6} + O(w^3)\right). \tag{C.8}$$

The remaining factor is regular at $v = -n$ and satisfies

$$\frac{\Gamma(1 - pv)\kappa^{-v}\sin(Pv)}{v} = r_{0,n} + r_{1,n}w + O(w^2), \tag{C.9}$$

where

$$r_{0,n} = -\frac{G_n S_n}{n},$$

$$r_{1,n} = -\frac{G_n}{n}\left(\left[-p\psi(1 + pn) - L\right]S_n + PC_n\right) - \frac{G_n S_n}{n^2}.$$

Consequently the first-derivative collision kernel

$$\frac{\Gamma(1 - pv)\Gamma(qv)\sin(Pv)\kappa^{-v}}{v\sin(\pi v)}$$

has the Laurent expansion

$$-D_n w^{-2} - I_n^{(1)} w^{-1} + O(1), \tag{C.10}$$

where D_n and $I_n^{(1)}$ are precisely the quantities in (B.9) *and* (B.8)*. More explicitly,*

$$D_n = -\frac{(-1)^{(q+1)n}}{\pi q (qn)!} r_{0,n},$$

$$I_n^{(1)} = -\frac{(-1)^{(q+1)n}}{\pi (qn)!}\left(\frac{r_{1,n}}{q} + h_{qn} r_{0,n}\right).$$

Moreover,

$$\psi(-v) + \gamma_E = H_{n-1} - \psi_1(n) w + O(w^2). \tag{C.11}$$

Thus, under the left-residue sign convention used in Theorem B.1*, the second-derivative integer contribution is*

$$I_n^{(2)} = 2\left(H_{n-1} I_n^{(1)} - \psi_1(n) D_n\right).$$

Proof. Equation (C.7) follows from (C.1) with $m = qn$ and $\varepsilon = qw$. Equation (C.8) follows from

$$\sin(\pi(-n + w)) = (-1)^n \left(\pi w - \frac{\pi^3 w^3}{6} + O(w^5)\right).$$

The factor in (C.9) is analytic at $v = -n$. Its value is $r_{0,n}$, and differentiating

$$\frac{\Gamma(1 - pv)\kappa^{-v}\sin(Pv)}{v}$$

at $v = -n$ gives $r_{1,n}$. Multiplication of (C.7), (C.8), and (C.9) yields

$$\frac{(-1)^{(q+1)n}}{\pi (qn)!}\left(\frac{r_{0,n}}{qw^2} + \frac{r_{1,n}/q + h_{qn} r_{0,n}}{w} + O(1)\right),$$

which is (C.10) after substituting the displayed definitions of D_n and $I_n^{(1)}$. Since $-v = n - w$, the Taylor expansion of $\psi(n - w) + \gamma_E$ gives (C.11). Multiplying (C.10) by (C.11) and applying the same left-residue sign convention gives the stated expression for $I_n^{(2)}$. ☐

D Canonical logit-atlas residual recurrences

This appendix supports Theorem III.15 and the certificate discussion in Subsection III-F. It records the residual D-algebraic system, the endpoint-matched compactification identities, and the coefficient recurrences on dyadic logit cells. The closure of D-algebraic functions under the operations used below is standard in the modern algorithmic treatment of D-algebraic functions; see [3], [4].

Lemma D.1 (Residual D-algebraic system)**.** *Let* $\delta = z\,\mathrm{d}/\mathrm{d}z$*, let* $G = \delta H_\rho$*, and let* $r_B = \delta B_\rho / B_\rho$*. Put*

$$Z_1 = H_\rho \log N_\rho, \qquad Z_2 = H_\rho (\log N_\rho)^2.$$

Then

$$H_\rho \delta Z_1 = G Z_1 + (G - r_B H_\rho) H_\rho, \tag{D.1}$$

$$H_\rho \delta Z_2 = G Z_2 + 2(G - r_B H_\rho) Z_1. \tag{D.2}$$

Moreover

$$H_\rho \log H_\rho = H_\rho \log B_\rho + Z_1,$$

$$H_\rho (\log H_\rho)^2 = H_\rho (\log B_\rho)^2 + 2(\log B_\rho) Z_1 + Z_2.$$

Proof. Since $N_\rho = H_\rho / B_\rho$,

$$\delta \log N_\rho = \frac{G}{H_\rho} - r_B.$$

For $Z_1 = H_\rho \log N_\rho$,

$$\delta Z_1 = G \log N_\rho + H_\rho \left(\frac{G}{H_\rho} - r_B \right) = \frac{G}{H_\rho} Z_1 + G - r_B H_\rho.$$

Multiplying by H_ρ gives (D.1). The same calculation with $Z_2 = H_\rho (\log N_\rho)^2$ gives

$$\delta Z_2 = \frac{G}{H_\rho} Z_2 + 2 \left(\frac{G}{H_\rho} - r_B \right) Z_1,$$

which is (D.2). The reconstruction identities are obtained by expanding $\log H_\rho = \log B_\rho + \log N_\rho$. ☐

Lemma D.2 (Endpoint-matched compactification)**.** *Let*

$$y = \frac{\kappa_\rho z^p}{1 + \kappa_\rho z^p}.$$

Then $z \in (0, \infty)$ *is mapped bijectively to* $y \in (0,1)$*, and*

$$z = \kappa_\rho^{-1/p} \left(\frac{y}{1-y} \right)^{1/p}, \qquad \frac{\mathrm{d}z}{\mathrm{d}y} = \frac{1}{p} \kappa_\rho^{-1/p} y^{1/p-1} (1-y)^{-1/p-1}.$$

Furthermore

$$B_\rho(y) = H_{0,\rho} (1-y)^\nu, \qquad H_\rho(z)\,\mathrm{d}z = C_\rho y^{1/p-1} (1-y)^{1/q-1} N_\rho(y)\,\mathrm{d}y,$$

where $C_\rho = H_{0,\rho} / (p \kappa_\rho^{1/p})$*.*

Proof. Solving $y = \kappa_\rho z^p / (1 + \kappa_\rho z^p)$ gives the displayed inverse formula for z. Differentiation gives the Jacobian. Since $1 + \kappa_\rho z^p = (1-y)^{-1}$, the factor becomes $B_\rho(y) = H_{0,\rho}(1-y)^\nu$. Finally, $H_\rho = B_\rho N_\rho$ and

$$\nu - \frac{1}{p} - 1 = \left(\frac{1}{p} + \frac{1}{q} \right) - \frac{1}{p} - 1 = \frac{1}{q} - 1,$$

which gives the displayed density-weight identity. ☐

Lemma D.3 (Logit form of the residual integrals)**.** *Let*

$$t = \log \frac{y}{1-y}, \qquad y = \frac{e^t}{1+e^t}.$$

Then

$$W_\rho(t) = C_\rho e^{t/p} (1 + e^t)^{-\nu}, \qquad \Phi_\rho(t) = \log H_{0,\rho} - \nu \log(1 + e^t),$$

and the residual moments are exactly (III.38) *and* (III.39)*.*

Proof. The differential identity $\mathrm{d}y = y(1-y)\,\mathrm{d}t$ changes the weight in Lemma D.2 into

$$C_\rho y^{1/p}(1-y)^{1/q}\,\mathrm{d}t = C_\rho e^{t/p}(1+e^t)^{-1/p-1/q}\,\mathrm{d}t.$$

Because $\nu = 1/p + 1/q$, this is $W_\rho(t)\,\mathrm{d}t$. Also $1-y=(1+e^t)^{-1}$, hence $\Phi_\rho(t) = \log H_{0,\rho} - \nu\log(1+e^t)$. Substitution in (III.34)–(III.35) gives the two logit integrals. ☐

Lemma D.4 (Cell coefficient construction)**.** *On a Taylor-admissible cell* $J = [c_J - h_J, c_J + h_J]$*, write* $t = c_J + u$ *and expand* N_ρ*,* W_ρ*, and* Φ_ρ *as in* (III.45)*. The coefficients of* N_ρ *are obtained from*

$$N_\rho(c_J+u) = \frac{H_\rho(z(c_J+u))}{B_\rho(c_J+u)}$$

by ordinary Taylor division, with the t*-coordinate factor* $B_\rho(c_J+u) = H_{0,\rho}(1+e^{c_J+u})^{-\nu}$ *explicit and nonzero on the cell disk. Let* $P_{J,n}, Q_{J,n}$ *be obtained from Theorem III.11. Then the coefficient of* u^n *in the first residual integrand on this cell is* $C^{(1)}_{\rho,J,n}$ *from* (III.47)*, and the coefficient of* u^n *in the second residual integrand is* $C^{(2)}_{\rho,J,n}$ *from* (III.48)*.*

Proof. The first logit integrand is $W_\rho N_\rho \log N_\rho$. The coefficients of $N_\rho \log N_\rho$ are $P_{J,n}$ by definition, so multiplying by the W_ρ series gives the convolution (III.47).

The second logit integrand is

$$W_\rho\left[2\Phi_\rho N_\rho \log N_\rho + N_\rho(\log N_\rho)^2\right].$$

The coefficients of $N_\rho(\log N_\rho)^2$ are $Q_{J,n}$. Multiplication by Φ_ρ and by W_ρ gives the nested convolution in (III.48). ☐

Lemma D.5 (Cell moments)**.** *For a cell* $J = [c_J - h_J, c_J + h_J]$*,*

$$\Omega_{J,n} = \int_{-h_J}^{h_J} u^n\,\mathrm{d}u = \begin{cases} 0, & n \text{ odd},\\ \dfrac{2h_J^{n+1}}{n+1}, & n \text{ even}.\end{cases}$$

Consequently, if $G(u) = \sum_{n\ge 0} g_n u^n$ *converges uniformly on the cell, then*

$$\int_{-h_J}^{h_J} G(u)\,\mathrm{d}u = \sum_{n\ge 0} g_n \Omega_{J,n}.$$

For the canonical unit cell $h_J = 1/2$*, this reduces to* $\Omega_{J,n} = 1/(2^n(n+1))$ *for even* n *and* 0 *for odd* n*.*

Proof. Termwise integration gives the last identity. The odd moments vanish by symmetry. If n is even, then

$$\int_{-h_J}^{h_J} u^n\,\mathrm{d}u = 2\frac{h_J^{n+1}}{n+1}.$$

☐

Lemma D.6 (Endpoint normalization and summability)**.** *For the endpoint-matched hyperexponential factor,*

$$N_\rho(t) \to 1 \quad (t\to -\infty), \qquad N_\rho(t)\to 1 \quad (t \to +\infty).$$

Moreover

$$W_\rho(t) \sim C_\rho e^{t/p} \quad (t\to-\infty), \qquad W_\rho(t)\sim C_\rho e^{-t/q} \quad (t\to+\infty).$$

Thus the tails of the residual integrals are exponentially damped, up to the linear factor $\Phi_\rho(t) \sim -\nu t$ *in the mixed part of* $\mathcal{R}_{2,\rho}$*.*

Proof. The first two limits are the endpoint normalization $N_\rho(0) = N_\rho(1) = 1$ in the compactified y coordinate. The weight asymptotics follow immediately from

$$W_\rho(t) = C_\rho e^{t/p}(1+e^t)^{-\nu}.$$

As $t\to-\infty$, $1+e^t \to 1$, so $W_\rho(t)\sim C_\rho e^{t/p}$. As $t\to+\infty$, $1+e^t \sim e^t$, so

$$W_\rho(t) \sim C_\rho e^{t/p-\nu t} = C_\rho e^{-t/q}.$$

Finally $\Phi_\rho(t) = \log H_{0,\rho} - \nu\log(1+e^t)$ is asymptotic to a constant as $t\to-\infty$ and to $-\nu t + O(1)$ as $t\to+\infty$. Exponential damping dominates this linear growth. ☐

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